Multi-valued graphs in embedded constant mean curvature disks

by

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ABSTRACT

In this thesis, we discuss several results concerning constant mean curvature surfaces. In this thesis, we prove that an embedded constant mean curvature disk with Gaussian curvature large at a point contains a multi-valued graph around that point on the scale of $|A|^2$. This generalizes Colding and Minicozzi's result for minimal surfaces. Moreover, we provide examples of CMC surfaces containing arbitrary large multi-valued graphs.

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Readers: Dr. William Minicozzi, Dr. Joel Spruck

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Contents

List of Figures

1 Introduction

Minimal surfaces are defined as surfaces which are critical points for the area functional. It so happens that the mean curvature of a minimal surface, the average of the two principal curvatures, is identically zero. Surfaces which are critical points for the area functional under a volume constraint are instead called constant mean curvature (CMC) surfaces and, in fact, the average of the two principal curvatures is constant. Not only are there plenty of mathematical examples for both of these surfaces (for instance planes, helicoids and catenoids are minimal surfaces while spheres, cylinders and Delauney surfaces are non zero CMC surfaces), but they can easily be realized and observed in the real world. In nature, the shape of a soap film approximates with great accuracy that of a minimal surface while soap bubbles provide the analogous approximation for CMC surfaces. In other words, a soap bubble is the least-area surface that encloses the fixed volume inside.

In this thesis we prove the following statement which also appears in my paper [20, Theorem 0.1.],

> *Let Σ be a non zero CMC embedded disk with Gaussian curvature large at a point then Σ contains a multi-valued graph around that point on the scale of the norm squared of the second fundamental form.*

Somewhat imprecisely, to contain a multi-valued graph (Definition 4.1 in this thesis) means that M looks like a helicoid, Fig. 1.

The helicoid is clearly not a graph, nonetheless, each half of the helicoid minus the vertical axis can be viewed as a graph over the universal cover of the punctured plane and this is, roughly speaking, what it means to contain a multi-valued graph. The proof relies heavily on two things: Knowing when a large CMC embedded geodesic ball is stable, and once that is known, what that stability implies. The main result is the following:

Theorem 1.1. *Given* $N \in \mathbb{Z}_+$, $\omega > 1$ *and* $\varepsilon > 0$, *there exist* $C = C(N, \omega, \varepsilon) > 0$, $H > 0$ *and* $\bar{l} > 1$ *so:*
Let $\Sigma \subset \mathbb{R}^3$ *be an embedded and simply connected constant mean curvature equal to* h *surface. If* $|h| < \frac{H}{r_0}$ *and*

$$\sup_{\Sigma \cap B_{r_0 \bar{l}}(0)} |A|^2 \leq 4C^2 r_0^{-2} \text{ and } |A|^2(0) = C^2 r_0^{-2}$$

for some $r_0 > 0$, *then* Σ *(after a rotation) contains an* N*-valued graph* $\Sigma_g \subset \Sigma$ *over* $D_{\omega \bar{R}} \backslash D_{\bar{R}}$ *where* $\bar{R} < \frac{r_0}{\omega}$ *(with gradient* $\leq \varepsilon$ *and* $\text{dist}_\Sigma(0, \Sigma_g) \leq 4\bar{R}$*).*

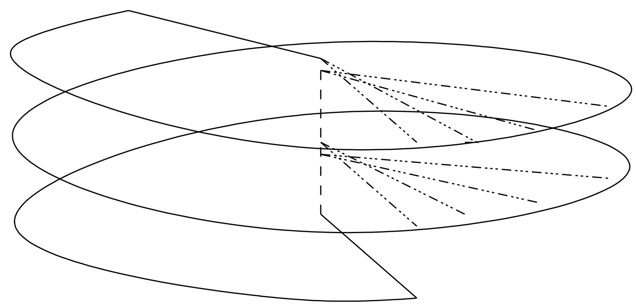

Figure 1: Half of the the helicoid

In the last chapter of this thesis we also give examples of non zero CMC surfaces containing arbitrary large multi-valued graphs. Using the method of successive approximations, a sequence of normal variations of the helicoid can be built which converges to a non zero CMC embedded disk containing a multi-valued graph.

1.1 Idea of the proof

The proof is by contradiction using a new compactness argument that does not require a bound on the area. The idea is the following: Assuming that Theorem 1.1 is false, we build a sequence Σ_n of embedded CMC disks where each disk satisfies the hypotheses of the theorem with C fixed large and $H = \frac{1}{n}$ but does not contain a N-valued graph. We prove that Σ_n converges to a minimal surface Σ_∞ which contains an N-valued graph. Essentially we show that Σ_n comes as close as we want to its limit and that the limit is an embedded minimal disk which contains an N-valued graph because of **Theorem 0.4.** in [4] (theorem 4.2 in this thesis), therefore so do the CMC surfaces. To create the N-valued graph in the CMC sequence we basically push the multi-valued graph from the minimal surface onto Σ_n. This contradiction proves the theorem.

The difficult part of the proof is to show that the limit is both an embedded surface and simply connected and not, for instance, a minimal lamination or a minimal surface which is not simply connected. Some sort of C^2 convergence follows in a standard way from the bound on the curvature and trivially, since we are assuming that the mean curvature goes to zero, the limit is a minimal object. The uniform bound on the curvature guarantees that in a sufficiently small ball the surface Σ_n looks like a finite collection of graphs. However, as n goes to infinity, it could happen that the number of graphs go to infinity. To assure that the limit surface is embedded and simply connected we need a uniform upper bound on the number of graphs.

After we investigate the strong stability for a constant mean curvature surface to find out when a CMC surface, which is already a critical point for a certain area functional, is an actual minimum, we proceed in three steps.

(i) We prove that, under certain conditions, if two CMC surfaces are close and disjoint, they are almost-stable.

(ii) We rule out the possibility that Σ_n contains a large, almost-stable domain ("almost a minimum"), for n large.

(iii) We show that if there is not a uniform upper bound on the number of pieces, then a large piece of Σ_n is a graph over another piece, creating a large almost-stable domain and giving the contradiction.

Once the uniform upper bound on the number of pieces is obtained, the convergence to an embedded minimal surface follows. We have to use some topological results to prove that the limit minimal surface is simply connected.

1.2 Sketch of the proof

We actually prove the result when r_0 in Theorem 1.1 is fixed and equal to one. The main result follows by rescaling. Fix $r_0 = 1$, assuming that Theorem 1.1 is false we have the following:

> Given $C(N, \omega, \varepsilon)$ as in **Theorem 0.4.** in [4], for any $h > 0$ there exists an embedded and simply connected constant mean curvature equal h surface Σ_h that does not contain an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega\bar{R}} \backslash D_{\bar{R}}$ for any $\bar{R} < \frac{1}{\omega}$ but such that
>
> $$0 \in \Sigma \subset B_{\bar{l}}(0) \subset \mathbb{R}^3,\ \partial\Sigma \subset \partial B_{\bar{l}}(0) \text{ and } \sup_{\Sigma\cap B_{\bar{l}}} |A|^2 \leq 4C^2 = 4|A|^2(0).$$

We want to show that this cannot be true. Let us take a sequence of Σ_n as above with $h = \frac{1}{n}$. The constant mean curvature of Σ_n goes to zero but none of the elements in the sequence contain an N-valued graph. Fixed $\bar{\varepsilon} > 0$, we consider a new sequence Σ'_n where Σ'_n is the connected component of $\Sigma_n \cap B_{\bar{l}-\bar{\varepsilon}}(0)$ that contains 0. Given that $|A|^2$ is bounded and we are slightly away from the boundary there exists $r > 0$ so: Σ'_n can be covered by a finite number of balls, $B_r(x_i^n)$ where $x_i^n \in \Sigma'_n$, such that in each ball $\Sigma'_n \cap B_r(x_i^n)$ looks like graphs u_n^j over the tangent plane $T_{x_i^n}\Sigma_n$. The radius r and the number of balls will be independent of n. Going to a subsequence, we can assume that x_i^n converges to a certain x_i and that $T_{x_i^n}\Sigma_n$ converges to a certain $T_{x_i}\Sigma_\infty$. At this point we are able to extract, by using Arzela-Ascoli, a subsequence u_n^j that converges uniformly to a graph u_∞^j. These CMC graphs satisfy the following partial differential equation

$$\frac{1}{n} = \text{div}\left(\frac{\nabla u_n^j}{\sqrt{1 + |\nabla u_n^j|^2}}\right).$$

Therefore, using Schauder theory [9] and the fact that $\frac{1}{n}$ goes to zero, we can prove that u_n^j converges C^2 to u_∞^j and that the latter is a minimal graph.

Unfortunately, we need more to prove the global properties required. The limit object contains a multi-valued graph if it is an embedded and simply connected minimal surface. We have not ruled out the possibility that the number j of graphs

u_n^j goes to infinity as n goes to infinity and in the limit that could give an infinite number of minimal graphs. As a consequence the limit would not necessarily be a surface but it could be a lamination. Another possibility is that the limit is not simply connected, for instance it could be a catenoid, Fig. 2. Rescaling the catenoid in Fig. 2

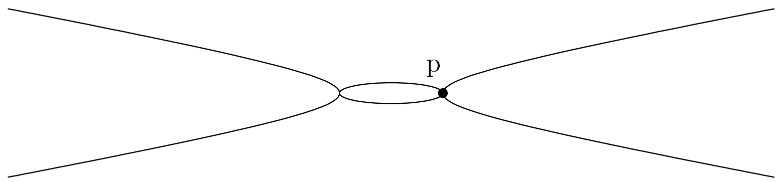

Figure 2: Catenoid

the curvature at p becomes very large and yet the catenoid would not contain a multi-valued graph. What we show is that the number of graphs is uniformly bounded if

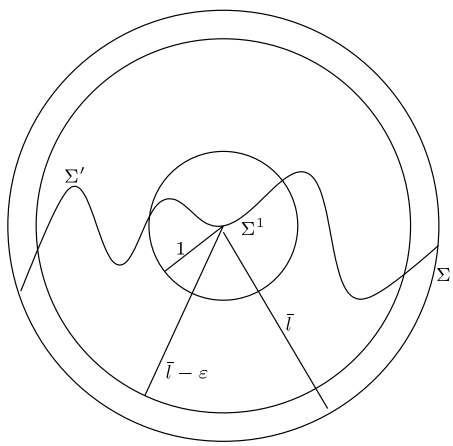

Figure 3: Σ, Σ', and Σ^1

we stay substantially away from the boundary. This is because to prove this uniform upper-bound on the number of graphs we have to work with large geodesic balls and to assure that they exist, we need to move substantially away from the boundary. We need to be working in the unit ball and keep the boundary of the surface on a substantially bigger ball. More precisely, we build another subsequence Σ_n^1 where Σ_n^1 is the connected component of $\Sigma_n \cap B_1(0)$ that contains 0, Fig. 3. Σ_n^1 is also simply connected. If it was not simply connected there would exist $B_r(0)$, $1 < r < \bar{l}$ such that Σ_n is tangent to $\partial B_r(0)$ and locally inside $B_r(0)$. This is a contradiction for

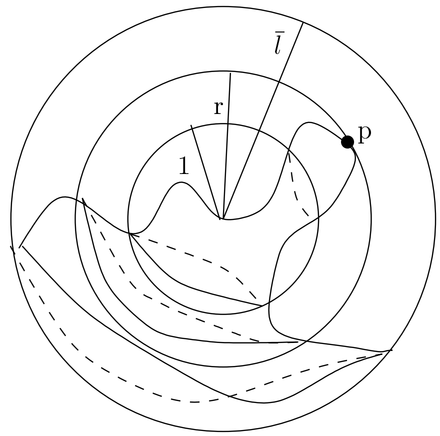

Figure 4: Simply Connected

$H(n) < \frac{1}{2l}$, Fig. 4. If we restrict our attention to Σ_n^1 we have a uniform upper-bound on the number of graphs and it follows that Σ_n^1, not the whole Σ_n, converges to an embedded minimal disk. Once we have that Σ_n^1 converges to an embedded minimal disk, we prove that Σ_n^1, and therefore Σ_n, contains a multi-valued graph.

1.3 Motivation and Future Directions

The work on this thesis targets mainly three objectives:

(i) It generalizes Colding and Minicozzi's result for minimal surfaces [4, Theorem 0.4.].

(ii) It is a first step towards a classification of singularities for sequences of embedded CMC disks; indeed, much more needs to be done in this direction.

(iii) The proof by contradiction provides a new type of compactness argument that does not require a bound on the area.

The work on this thesis has left open a few questions:

(i) In the minimal case Colding and Minicozzi were able to extend the multi-valued graph that forms locally, all the way up to the boundary [3]. Is it possible for non zero CMC embedded disks to extend the multi-valued graph to a larger scale?

(ii) In the minimal case Colding and Minicozzi were able to prove that an embedded minimal disk can be broken up into two types of building blocks [6]. Where the Gaussian curvature is small we have graphical pieces, where it is large it looks like a helicoid. The helicoid is, of course, one example and Matthias Weber has recently given many others. Can the same global structure theorem be proved for non zero CMC embedded disks?

(iii) Is it possible to prove global compactness results for complete non zero CMC embedded surfaces? More precisely, given a sequence of CMC embedded surfaces, is it true that there exists a subsequence converging to a CMC lamination?

(iiii) Can the same result be proven for surfaces with small mean curvature, not necessarily constant?

2 Background on Constant Mean Curvature Surfaces

This chapter is a short review of constant mean curvature surfaces and their properties. In the first section we set the notation and give the definition of mean curvature. In the second section we analyze constant mean curvature graphs. In the third section we prove a few standard results about CMC surfaces and stability. In the fourth section we study under what conditions and in what sense we can take limits of a given sequence of surfaces.

2.1 The First Variation Formula; Mean Curvature

Let $\Sigma \subset \mathbb{R}^3$ be a 2-dimensional smooth orientable surface (possibly with boundary) with unit normal N_Σ. Given a function ϕ in the space $C_0^\infty(\Sigma)$ of infinitely differentiable (i.e., smooth), compactly supported functions on Σ, consider the one-parameter variation

$$\Sigma_{t,\phi} = \{x + t\phi(x)N_\Sigma(x) | x \in \Sigma\}$$

and let $A(t)$ be the area functional,

$$A(t) = Area(\Sigma_{t,\phi}).$$

The so-called first variation formula of area is the equation (integration is with respect to $d$area)

$$A'(0) = \int_\Sigma \phi H, \tag{2.1}$$

where H is the mean curvature of Σ. When H is constant the surface is said to be a *constant mean curvature* (CMC) surface [14] and it is a critical point for the area functional restricted to those variations which preserve the *enclosed volume*, in other words ϕ must satisfy the condition,

$$\int_\Sigma \phi = 0.$$

In general, let k_1, k_2 be the principal curvatures on Σ, then $H = k_1 + k_2$; $|A|^2 = k_1^2 + k_2^2$ is the norm squared of the second fundamental form. Since the Gaussian curvature K_Σ

is equal to the product of the principal curvatures k_1k_2, we have the Gauss equation, that is

$$H^2 = |A|^2 + 2K_\Sigma. \tag{2.2}$$

From (2.2) it is clear why when H is constant, in particular when it is small and even better when it is zero, talking about the Gaussian curvature or talking about the norm of the second fundamental form squared is almost equivalent.

Concrete examples of constant mean curvature surfaces are spheres, cylinders and Delauney surfaces. In the particular case where the mean curvature H is identically zero the surface Σ is said to be a *minimal* surface [16, 2].

2.2 Constant Mean Curvature Graphs

If Σ is a surface given as a graph of a function u then

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{2.3}$$

Therefore, when H is constant u satisfies a quasi-linear differential equation.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be an open connected neighborhood of the origin. Let u_1, u_2 be CMC graphs over Ω and assume that they have the same constant mean curvature ($|H_{u_1}| = |H_{u_2}|$), and the same orientation ($\langle N_1, N_2\rangle > 0$). If $u_1 \leq u_2$ and $u_1(0) = u_2(0)$, then $u_1 \equiv u_2$.*

Proof. The proof is a slight modification of Lemma 1.17 [4, p.15]. Since

$$\begin{aligned}
&\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}} - \frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}} \\
&\quad= \frac{\sqrt{1+|\nabla u_2|^2}\,(\nabla u_1 - \nabla u_2) + (\sqrt{1+|\nabla u_2|^2} - \sqrt{1+|\nabla u_1|^2})\,\nabla u_2}{\sqrt{1+|\nabla u_1|^2}\,\sqrt{1+|\nabla u_2|^2}} \\
&\quad= \frac{\nabla u_1 - \nabla u_2}{\sqrt{1+|\nabla u_1|^2}} + \frac{(|\nabla u_2|^2 - |\nabla u_1|^2)\,\nabla u_2}{(\sqrt{1+|\nabla u_1|^2} + \sqrt{1+|\nabla u_2|^2})\,\sqrt{1+|\nabla u_1|^2}\,\sqrt{1+|\nabla u_2|^2}}
\end{aligned}$$

and both u_1 and u_2 satisfy the minimal surface equation, we get

$$\begin{aligned} 0 &= \operatorname{div}\left(\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}} - \frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}}\right) \\ &= \operatorname{div}\left(\frac{\nabla(u_1-u_2)}{\sqrt{1+|\nabla u_1|^2}}\right) \\ &\quad - \operatorname{div}\left(\frac{< \nabla(u_1-u_2), \nabla(u_1+u_2) >}{(\sqrt{1+|\nabla u_1|^2}+\sqrt{1+|\nabla u_2|^2})\sqrt{1+|\nabla u_1|^2}\sqrt{1+|\nabla u_2|^2}}\nabla u_2\right). \end{aligned}$$

From this, we conclude that $v = u_1 - u_2$ satisfies an equation of the form

$$0 = \operatorname{div}(a_{i,j}\nabla v) + b_i \nabla v\,.$$

Moreover, if $|\nabla u_1|, |\nabla u_2|$ are sufficiently small, then $\lambda|x|^2 \leq a_{i,j}x_ix_j$ for some $\lambda > 0$. Therefore, the usual strong maximum principle gives the claim. □

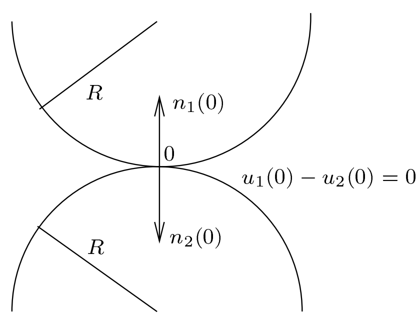

Figure 5: Harnack Inequality

As a consequence we have the following Harnack type inequality

Corollary 2.2. *Let u_1, u_2 be CMC graphs over $D_r(0)$ and assume that they have the same constant mean curvature ($H_{u_1} = H_{u_2}$), the same orientation ($\langle N_1, N_2\rangle > 0$), and that $u_1 - u_2 > 0$ if $|\nabla u_1|$ and $|\nabla u_2|$ are sufficientely small*

$$\sup_{B_{\frac{r}{2}}(0)} (u_1 - u_2) \leq C_0(u_1(0) - u_2(0)). \tag{2.4}$$

Notice from Fig. 5 that the condition $\langle N_1, N_2\rangle > 0$ on the orientation is necessary. As it is shown in Fig. 5 the two spherical caps have the same constant mean curvature since they have the same radius. However, even if $u_1(0) - u_2(0) = 0$, it is clear that $\sup(u_1 - u_2) > 0$ in any neighborhood of 0 and therefore that (2.4) does not follow.

2.3 The Second Variation Formula; Stability

Let A be the area functional described in Section 2.1; we showed that $A'(0) = \int_\Sigma \phi H$. A computation shows that if Σ is a CMC surface then

$$A''(0) = -\int_\Sigma \phi L_\Sigma \phi, \qquad \text{where } L_\Sigma \phi = \Delta_\Sigma \phi + |A|^2 \phi \tag{2.5}$$

is the second variational operator. Here Δ_Σ is the intrinsic Laplacian on Σ. A CMC surface Σ is said to be (strongly) stable if

$$A''(0) \geq 0 \qquad \text{for all } \phi \in C_0^\infty(\Sigma). \tag{2.6}$$

Applying Stokes' theorem to (2.6) shows that Σ is stable if and only if

$$\int_\Sigma |A|^2 \phi^2 \leq \int_\Sigma |\nabla \phi|^2, \qquad \text{for all } \phi \in C_0^\infty(\Sigma)$$

and that allows us to define δ-stability, namely Σ is said to be δ-stable if

$$(1-\delta) \int_\Sigma |A|^2 \phi^2 \leq \int_\Sigma |\nabla \phi|^2, \qquad \text{for all } \phi \in C_0^\infty(\Sigma). \tag{2.7}$$

In the following lemma we establish a relation between a CMC surface and a CMC normal variation of it that does not change the mean curvature.

Lemma 2.3. *There exists $\delta_1 > 0$ so: If $\delta < \delta_1$, Σ is a CMC surface and u is a positive solution of the CMC graph equation over Σ (i.e. $\Sigma^u := \{x + u(x)N_\Sigma(x) | x \in \Sigma\}$ is CMC) such that $|H_{\Sigma^u}| = |H_\Sigma|$, $\langle N_{\Sigma^u}, N_\Sigma \rangle \geq 0$, $|u||A|$ and $|\nabla u| \leq \delta$ then $\triangle u + u|A|^2 = o(\delta^2)$.*

Proof. In general, see **??** and **??**

$$H_{\Sigma^u} = H_\Sigma + \frac{1}{2}(\triangle u + u|A|^2) + o(|u|^2, |\nabla u|^2).$$

The condition $\langle N_{\Sigma^u}, N_\Sigma \rangle \geq 0$ is a condition on the orientation that implies $H_{\Sigma^u} = H_\Sigma$ and the lemma follows. $\square$

The existence of a positive solution of $Lu = 0$ where L is $\Delta + |A|^2$ would imply $A''(0) \geq 0$ for all $\phi \in C_0^\infty(\Sigma)$. In the following lemma we show that if there exists a positive function u which is "almost" a solution, then $A''(0)$ is "almost" non-negative for all $\phi \in C_0^\infty(\Sigma)$, that is, almost-stable.

Lemma 2.4. *Let Ω be a domain and u be a positive function in $C^2(\Omega)$ such that*

$$\Delta u \leq -(1-\delta)|A|^2 u \tag{2.8}$$

then Ω is δ-stable.

Proof. Set $w = \log u$ and let Φ be any compactly supported function on Ω. We have

$$\operatorname{div}(\nabla w) = \operatorname{div}(\frac{\nabla u}{u}) = \frac{\triangle u}{u} - \frac{|\nabla u|^2}{u^2} = \frac{\triangle u}{u} - |\nabla w|^2.$$

Applying Stokes theorem to $\operatorname{div}(\Phi^2 \nabla w)$ gives

$$0 = \int \operatorname{div}(\Phi^2 \nabla w) = \int \Phi^2 \Delta w + \int \langle \nabla \Phi^2, \nabla w \rangle.$$

Using Cauchy-Shwarz and the absorbing inequality gives

$$\int \langle \nabla \Phi^2, \nabla w \rangle \leq \int |\nabla \Phi|^2 + \int \Phi^2 |\nabla w|^2.$$

Eventually,

$$\int (-\frac{\triangle u}{u} + |\nabla w|^2)\Phi^2 \leq \int |\nabla \Phi|^2 + \int \Phi^2 |\nabla w|^2.$$

Applying (2.8) we get

$$(1-\delta)\int_\Sigma |A|^2 \phi^2 \leq \int_\Sigma |\nabla \phi|^2.$$

□

Lemma 2.3 and Lemma 2.4 give a first criteria to find almost stable domains in a constant mean curvature surface.

Corollary 2.5. *There exists $\delta_3 > 0$ so: If $\delta < \delta_3$, Σ is a CMC surface and u is a positive solution of the CMC graph equation over Σ such that $|H_{\Sigma^u}| = |H_\Sigma|$, $\langle N_{\Sigma^u}, N_\Sigma \rangle \geq 0$, $|u||A|$ and $|\nabla u| \leq \delta$ then Σ is δ-stable.*

2.4 Limits of surfaces

This material is covered in great detail (including proofs) in [17, Section 4].

Let Σ be a surface in $\mathbb{R}^3$ with tangent plane $\mathrm{T}_x\Sigma, x \in \Sigma$. Given $x \in \Sigma$ we label by

$$D(x,r) = \{x + v | v \in \mathrm{T}_x\Sigma, |v| < r\} \text{ the tangent disk of radius } r.$$

$W(x,r)$ stands for the infinite solid cylinder of radius r around the affine normal line at x,

$$W(x,r) = \{q + tN(x) | q \in D(x,r), t \in \mathbb{R}\}.$$

Inside $W(x,r)$ and for $\varepsilon > 0$, we have the compact slice

$$W(x,r,\varepsilon) = \{q + tN(x) | q \in D(x,r), |t| < \varepsilon\}.$$

Definition 2.6. *A sequence Σ_n of surfaces converges to a set Σ_∞ in the C^k topology with finite multiplicity if Σ_∞ is the accumulation set of Σ_n and for all $p \in \Sigma_\infty$ there exist an r and ε such that:*

(i) *$W(p,r,\varepsilon) \cap \Sigma_\infty$ can be expressed as the graph of a function $u : D(p,r) \to \mathbb{R}$.*

(ii) *For n large enough $W(p,r,\varepsilon) \cap \Sigma_n$ consists of a finite number (independent of n) of graphs over $D(p,r)$ which converge to u in the C^k topology.*

The following is a compactness theorem for properly embedded minimal surfaces with norm of the second fundamental form uniformly bounded.

Theorem 2.7. *Let $O_n \subset O_{n+1}$ be a sequence of open sets and Σ_n be a minimal surface properly embedded in O_n. Suppose that there exists a sequence $p_n \in \Sigma_n$ converging to a point $p \in O_\infty$ and that $\sup_{\Sigma_n} |A_n|^2 < C$ uniformly. Then, there exists a subsequence Σ_{n_k} and a connected minimal surface Σ in O_∞ satisfying*

(i) *Σ is contained in the accumulation set of Σ_n.*

(ii) *$p \in \Sigma$ and $|A|(p_n) = \lim |A_n|(p_n)$.*

(iii) *Σ is embedded in O_∞ (but not necessarily properly embedded).*

(iiii) *Any divergent path in Σ either diverges in O or has infinite length.*

Corollary 2.8. *If in Theorem 2.7 we assume that $O_\infty = \mathbb{R}^3$ then Σ is complete.*

Proof. Property (iiii) of Theorem 2.7 says that any divergent path in Σ either diverges in $\mathbb{R}^3$ or has infinite length. However, a divergent path in $\mathbb{R}^3$ has infinite therefore any divergent path in Σ has infinite length that is Σ is complete. $\square$

Theorem 2.7 and Corollary 2.8 hold even if, instead of having a sequence of minimal surfaces with uniform bound on the second fundamental form, we have a sequence of surfaces with uniform bound on the second fundamental form with mean curvature going to zero.

3 Curvature Estimates

In this chapter we want to prove a curvature estimate for stable surfaces with bounded mean curvature embedded in $\mathbb{R}^3$. In particular, we intend to generalize the following well-know curvature estimate for stable minimal surfaces due to Schoen [18] to surfaces embedded in $\mathbb{R}^3$ with bounded mean curvature.

Theorem 3.1. *If $\Sigma^2 \subset M^3$ is an immersed stable minimal surface with trivial normal bundle and $B^{\Sigma}_{r_0}(x) \subset \Sigma \backslash \partial\Sigma$, where $|K_M| \leq k^2$ and $r_0 < \rho_1(\frac{\pi}{k}, k^2)$, then for some $C(k)$ and all $0 < \sigma \leq r_0$,*

$$\sup_{B^{\Sigma}_{r_0-\sigma}} |A|^2 \leq C\sigma^{-2}.$$

For CMC surfaces the generalization was proved by Sirong Zhang in [21, **Theorem 0.1.**]. Here we give a different proves that uses a compactness argument.

3.1 Curvature Estimates for Stable Surfaces

In [21, **Theorem 0.1.**] Sirong Zhang proved the following,

Theorem 3.2. *There exists a C such that given any $l > 0$ there exists an $h > 0$ so: If $\mathcal{B}_l(0)$ is a "constant mean curvature equal to h", δ-stable intrinsic disk with trivial normal bundle then*

$$\sup_{\mathcal{B}_{\frac{l}{2}}(0)} |A|^2 \leq \frac{C}{l^2}.$$

Theorem 3.2 can be thought as a generalization of [7, 8].

Here, we prove the same curvature estimate for extrinsic ball assuming only a bound on the mean curvature. We give a different proof that uses a compactness argument.

Theorem 3.3. *There exists a C so: Let $\Sigma \subset \mathbb{R}^3$ be an embedded stable surface such that $0 \in \Sigma \subset B_R(0)$, $\partial\Sigma \subset \partial B_R(0)$, and $|H| < \frac{1}{R}$ then*

$$\sup_{\Sigma}(R - |x|)^2|A|^2(x) \leq C.$$

Proof. The proof by contradiction uses a compactness argument. Let $f(x) = (R - |x|)^2|A|^2(x)$, we want to show that $max_{\Sigma} f(x) < C$. Assume that the theorem is false, then for any $n \in \mathbb{N}$ there exists an embedded stable surface Σ_n such that $0 \in \Sigma_n \subset B_R(0)$, $\partial\Sigma_n \subset \partial B_R(0)$, $|H_n| < \frac{1}{R}$, but $max_{\Sigma_n} f_n(x) > n$. In other words, let x_n be the point in Σ_n where f_n obtain its maximum then,

$$(R - |x_n|)^2|A_n|^2(x_n) > n.$$

Notice that $|A|^2(x_n) > 0$ and that

$$R_n'^2 = (R - |x_n|)^2 > \frac{n}{|A_n|^2(x_n)}$$

. Let's center the connected component of $B_{R_n'}(x_n) \cap \Sigma_n$ containing x_n at the origin. We now have a new sequence of surfaces, that we shall call Σ_n, so: $0 \in \Sigma_n \subset B_{R_n'}(0)$, $\partial\Sigma_n \subset \partial B_{R_n'}(0)$, $|H_n| < \frac{1}{R}$, ${R'_n}^2 > \frac{n}{|A_n|^2(0)}$, and $|A|(0) > 0$. Let's consider another sequence, Σ_n', obtained by rescaling each Σ_n by a factor $|A_n|(0)$. In this way, we have a new sequence so: $0 \in \Sigma_n' \subset B_{|A_n|(0)R_n'}(0)$, $\partial\Sigma_n' \subset \partial B_{|A_n|(0)R_n'}(0)$, $|H_n| < \frac{1}{R|A_n|(0)}$, and $|A_n'|(0) = 1$. Notice that

$$R^2|A_n|^2(0) > R_n'^2|A_n|^2(0) > n,$$

in other words the radius of the ball goes to infinity as n goes to infinity, while H_n goes to zero. Theorem 2.7 and Corollary 2.8 give that Σ_n' converges to a complete embedded minimal surface Σ such that $|A|^2(0) = 1$. If we prove that Σ is stable then Σ must be a plane [7, 8], and that contradicts $|A|^2(0) = 1$. Let's assume Σ is not stable. Then, there exists $K \subset \Sigma$ compact set and $\Phi \in C_0^\infty(K)$ such that

$$\int_K |A|^2\Phi^2 > \int_K |\nabla\Phi|^2 + \varepsilon, \quad \varepsilon > 0.$$

Let $K_n \subset \Sigma_n$ be a sequence of compact sets converging to K with multiplicity one and let's define $\Phi_n \in C_0^\infty(K_n)$ so: $\Phi_n(x_n) = \Phi(x)$ where $x = \lim_n x_n$. Taking n big enough, we have

$$\int_K |A|^2\Phi^2 < \int_{K_n} |A_n|^2\Phi_n^2 + \frac{\varepsilon}{2} \text{ and } \int_K |\nabla\Phi|^2 > \int_{K_n} |\nabla\Phi_n|^2 - \frac{\varepsilon}{2}.$$

Eventually, we have

$$\int_{K_n} |A_n|^2\Phi_n^2 > \int_{K_n} |\nabla\Phi_n|^2$$

which is a contradiction because we are assuming Σ_n stable. $\square$

4 Multi-valued graph in constant mean curvature embedded disk

This is the definition of multi-valued graph:

Definition 4.1 (Multi-valued graph). *Let D_r be the disk in the plane centered at the origin and of radius r and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbb{C}\backslash 0$ with global coordinates (ρ, θ) so $\rho > 0$ and $\theta \in \mathbb{R}$. An N-valued graph of a function u on the annulus $D_s \backslash D_r$ is a single valued graph over $\{(\rho, \theta) | r \leq \rho \leq s, |\theta| \leq N\pi\}$.*

When dealing with multi-valued graphs, the surface to keep in mind is the helicoid, Fig. 1. A parametrization of the helicoid that illustrates the existence of such an N-valued graph is the following

$$(s \sin t, s \cos t, t) \qquad \text{where } (s,t) \in \mathbb{R}^2.$$

It is easy to see that it contains the N-valued graph ϕ defined by

$$\phi(\rho, \theta) = \theta \qquad \text{where } (\rho, \theta) \in \mathbb{R}^+ \backslash 0 \times [-N\pi, N\pi].$$

This is what Colding and Minicozzi proved:

Theorem 4.2. *[4, **Theorem 0.4.**] Given $N \in \mathbb{Z}_+$, $\omega > 1$ and $\varepsilon > 0$, there exist $C = C(N, \omega, \varepsilon) > 0$ so:*
Let $0 \in \Sigma \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk such that $\partial\Sigma \subset B_R$. If

$$\sup_{\Sigma \cap B_{r_0}} |A|^2 \leq 4C^2 r_0^{-2} \text{ and } |A|^2(0) = C^2 r_0^{-2}$$

for some $0 < r_0 < R$, then there exists $\bar{R} < \frac{r_0}{\omega}$ and (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega \bar{R}} \backslash D_{\bar{R}}$ with gradient $\leq \varepsilon$ and $\text{dist}_\Sigma(0, \Sigma_g) \leq 4\bar{R}$.

In this chapter we prove the main result, Theorem 1.1, with $r_0 = 1$. That is,

Theorem 4.3. *For each $N \in \mathbb{Z}_+$, $\omega > 1$ and $\varepsilon > 0$ there exist $H > 0$, $C(N, \omega, \varepsilon) > 0$ and $\bar{l} > 1$ so:*

Let $0 \in \Sigma \subset B_{\bar{l}}(0) \subset \mathbb{R}^3$ be an embedded and simply connected constant mean curvature surface equal to h (embedded CMC disk) such that $|h| \leq H$ and $\partial\Sigma \subset \partial B_{\bar{l}}(0)$. If

$$\sup_{\Sigma \cap B_{\bar{l}}(0)} |A|^2 \leq 4C^2 \text{ and } |A|^2(0) = C^2$$

then there exists $\bar{R} < \frac{1}{\omega}$ and (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega\bar{R}} \backslash D_{\bar{R}}$ (with gradient $\leq \varepsilon$ and $\text{dist}_\Sigma(0, \Sigma_g) \leq 4\bar{R}$).

The constant $C(N, \omega, \varepsilon)$ is essentially the same constant that Colding and Minicozzi used.

From now on, even if the results can often be stated more generally, Σ will be a CMC surface satisfying the hypotheses of Theorem 4.3 with . Σ' will be the connected component of $\Sigma \cap B_{\bar{l}-\bar{\varepsilon}}(0)$ containing 0. Σ^1 will be the connected component of $\Sigma \cap B_1(0)$ containing 0. We will also assume the mean curvature to be as small as we need, in particulary bounded.

In the first section we use the upper bound on $|A|^2$ to generalize some standard local results regarding CMC surfaces and we prove a criteria to find large pieces of Σ' which are graph over other pieces, creating large almost stable CMC domains. In the second section we show that an almost stable domain cannot be too large if it has to be contained in Σ^1 and how this implies a uniform upper bound on the number of graphs. In the third we prove that Σ^1_n converges C^2 to an embedded minimal disk Σ_∞ that contains a multi-valued graph. It follows that the CMC surfaces in the sequence contain a multi-valued graph as well.

4.1 The upper bound on $|A|^2$

Let us define

$$\Sigma_{x,R} \text{ as the component of } B_R(x) \cap \Sigma \text{ that contains } x, \tag{4.1}$$

$$\mathcal{B}_R(x) := \{y \in \Sigma \text{ such that } \text{dist}_\Sigma(y, x) < R\} \tag{4.2}$$

i.e., the geodesic ball of radius R centered at x,

$$\mathcal{D}_r(x) := \{x' \in T_x\Sigma \text{ such that } |x - x'| < r\}. \tag{4.3}$$

In what follows we are about to explain why in a CMC surface with bounded $|A|^2$ everything looks graphical–what we have been calling "uniformly locally flat." In general, let Σ be a surface with bounded bounded mean curvature and bounded second fundamental form, integrating $|\nabla \text{dist}_{S^2}(\mathbf{n}(x), \mathbf{n})| \leq |A|$ on geodesics gives

$$\sup_{x' \in \mathcal{B}_s(x)} \text{dist}_{S^2}(\mathbf{n}(x), \mathbf{n}) \leq s \sup_{\mathcal{B}_s(x)} |A|. \tag{4.4}$$

By (4.4), we can choose $0 < \rho < \frac{1}{4}$ so: If $\mathcal{B}_{2s}(x) \subset \Sigma$, $s \sup_{\mathcal{B}_s(x)} |A| \leq 4\rho_2$, and $t \leq s$ then the component $\Sigma_{x,t}$ of $B_t(x) \cap \Sigma$ with $x \in \Sigma_{x,t}$ is a graph over $T_x\Sigma$ with gradient $\leq \frac{t}{s}$ and

$$1 \geq \inf_{x' \in \mathcal{B}_2 s(x)} \frac{|x' - x|}{\text{dist}_\Sigma(x, x')} > \frac{9}{10}. \tag{4.5}$$

One consequence is that if $t \leq s$, then

$$\sup_{x' \in \mathcal{B}_t(x)} |x' - D(x,t)| \leq \frac{t^2}{s}. \tag{4.6}$$

As a consequence of (4.4), (4.5), (4.6) and the fact that $\sup_\Sigma |A| < C$ we can clearly choose $0 < \bar{\rho} < 4\frac{\rho}{C}$ so: Given $t < \bar{\rho}$ and $x \in \Sigma$ then

$$\Sigma_{x,t} \text{ is a graph over } T_x\Sigma \text{ with gradient } \leq \frac{t}{\bar{\rho}} \text{ and } 1 \geq \inf_{x' \in \mathcal{B}_{2\bar{\rho}}(x)} \frac{|x' - x|}{\text{dist}_\Sigma(x, x')} > \frac{9}{10}.$$

This means that, independently on x, $\Sigma_{x,t}$ is a graph over $T_x\Sigma$. Moreover, as shown in Fig. **??**, using the Pythagorean theorem gives that

$$\text{the projection of } \Sigma_{x,t} \text{ onto } T_x\Sigma \text{ contains } \mathcal{D}_{\sqrt{t^2 - \frac{t^4}{\bar{\rho}^2}}}(x). \tag{4.7}$$

Furthermore, if $y \in B_t(x) \cap \Sigma$ and $\text{dist}_\Sigma(x, y) \geq 2t$ then y cannot be in $\Sigma_{x,t}$, otherwise applying (4.5) gives

$$t \geq |y - x| > \frac{9}{10}\text{dist}_\Sigma(x, y) \geq \frac{18}{10}t. \tag{4.8}$$

y is in a different component of $B_t(x) \cap \Sigma$. After defining an orientation y is either above or below $\Sigma_{x,t}$. For the same reason we can also add that $\mathcal{B}_{\bar{\rho}}(x) \cap \mathcal{B}_{\bar{\rho}}(y) = \emptyset$.

Corollary 2.5 tells us that under certain conditions regarding the orientation, if a CMC surface is a graph over another CMC surface with the same constant mean curvature, then it is almost stable. We are about to prove some lemmas which tell us when that happens and how large the almost stable domain is. This lemma shows how, if two pieces of Σ are close, then they must be graphs over the same plane.

Lemma 4.4. *There exists $\alpha_1 > 0$ so: For any $\alpha < \alpha_1$ and $x \in \Sigma'$ then any component of $B_\alpha(x) \cap \Sigma'$ is a graph over $T_x\Sigma$.*

Proof. Let us assume that there exists a component of $B_\alpha(x) \cap \Sigma'$ which is not a graph over $T_x\Sigma$. Then there exists $y \in B_\alpha(x) \cap \Sigma'$ such that $T_x\Sigma \perp T_y\Sigma$. If α_1 is small enough, it is clear from Fig. 6 that $\Sigma_{x,\frac{\bar{\rho}}{2}} \cap \Sigma_{y,\frac{\bar{\rho}}{2}} \neq \emptyset$ that is $y \in \mathcal{B}_{\bar{\rho}}(x)$. How we

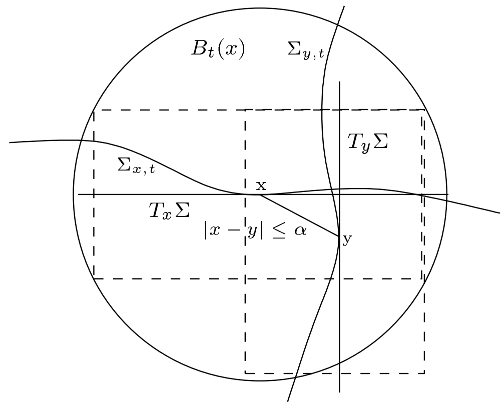

Figure 6: Graphical Pieces

have chosen $\bar{\rho}$ implies that y must be part of a graph. Notice that we are also using the fact that we are slightly away from the boundary. $\Sigma_{x,\frac{\bar{\rho}}{2}} \cap \Sigma_{y,\frac{\bar{\rho}}{2}}$ could be empty if one of the two sets reaches $\partial\Sigma$ before they intersect. How small α_1 must be will depend also on $\bar{\varepsilon}$.

□

In particular, it follows that if pieces of Σ' are very close then not only are they graphs over the same plane, they are graphs over each other. The idea is that if two graphs are almost flat over two different planes but they cannot intersect, then if the two graphs are close enough these two planes must have almost the same slope. One of the two graphs can therefore be seen as a graph over the other and this is what the next lemma is about.

Lemma 4.5. *There exists $\alpha_2 > 0, C_1 > 0$ and $s > 0$ so: Let $x, y \in \Sigma'$ such that $|x-y| \leq \alpha < \alpha_2$, $d_\Sigma(x,y) \geq 2\alpha$ and $\langle n(x), n(y)\rangle > 0$ then $\mathcal{B}_s(y)$ contains a graph $\{z + u(z)n(z)\}$ over a domain containing $\mathcal{B}_{\frac{s}{4}}(x)$, $|\nabla u| + |u| \leq \alpha C_1$.*

Proof. Assume $\alpha_2 < \alpha_1$. We know from (4.8) that y is in a different component of $B_\alpha(x) \cap \Sigma'$ and that $\mathcal{B}_{\bar{\rho}}(x) \cap \mathcal{B}_{\bar{\rho}}(y) = \emptyset$. If α_2 is sufficiently small, it now follows that $\Sigma_{x,\bar{\rho}}$ and $\Sigma_{y,\bar{\rho}}$ contain two graphs over the same plane, the smaller α_2 is the bigger the graph is. There exists $s > 0$ such that $\mathcal{B}_s(x)$ and $\mathcal{B}_s(y)$ contain respectively a graph u_1 and u_2 over $\mathcal{D}_{\frac{s}{2}}(x)$. $\langle n(x), n(y)\rangle > 0$ implies that the two intrinsic disks have equal constant mean curvature and (2.4) gives $\sup_{\mathcal{D}_{\frac{s}{4}(x)}} |u_1 - u_2| \leq \alpha C_0$. The function

$$u(x) = \min\{t \in \mathbb{R}_+ | x + tN(x) \in \mathcal{B}_s(y)\} \tag{4.9}$$

is well defined over $B_{\frac{s}{4}}(x)$ and $|\nabla u| + |u| \leq \alpha C_1$. □

Lemma 2.4 and Lemma 4.5 give a better criteria to find δ-stable domain.

Corollary 4.6. *Given $\delta > 0$ there exists $\alpha_3 > 0$ and $s > 0$ so: Let $x, y \in \Sigma'$ such that $|x - y| \leq \alpha < \alpha_3$, $d_\Sigma(x, y) \geq 2\alpha$ and $\langle n(x), n(y)\rangle > 0$ then $\mathcal{B}_{\frac{s}{4}}(x)$ is δ-stable.*

Proof. Let $\alpha_3 < \min(\frac{\delta}{C_1}, \alpha_2)$ and apply Lemma 2.4 and Lemma 4.5. □

In sum, we have proven that when two points $x, y \in \Sigma'$ are close enough to each other (Euclidean distance) and satisfy the condition on the orientation $\langle N(x), N(y)\rangle > 0$, then a little neighborhood of each point is δ-stable. We shall notice that the closer two pieces are the smaller δ is. The next step is to go from a little almost stable domain to a large one. If we need a very large δ-stable geodesic ball, first of all we need

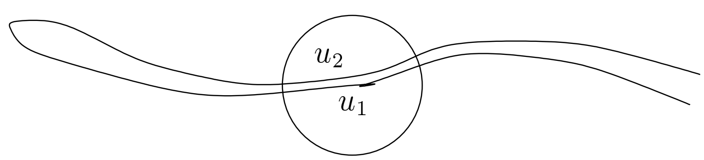

Figure 7: Harnack Inequality

the geodesic ball to be contained in Σ. In order to achieve this we certainly cannot be anywhere in the surface but sufficiently away from its boundary. If we move away from the boundary, as long as the objects we are working with are contained in Σ, thanks to the Harnack inequality we can find conditions that guarantee the existence of arbitrary large δ-stable domains. This is what we prove in the next lemma and corollaries. In Fig. 7 it is shown how if two pieces of Σ^1 are close then their extensions will have to stay relatively close.

Lemma 4.7. *For each* $0 < l < \bar{l} - (1+\bar{\varepsilon})$ *there exist* $\alpha_l > 0$ *and* $C_l > 0$ *so: Given* $\alpha < \alpha_l$ *and* $x, y \in \Sigma^1$ *such that* $|x-y| \le \alpha$, $d_\Sigma(x,y) \ge C_l$ *and* $\langle n(x), n(y)\rangle > 0$ *then for each* $x' \in \mathcal{B}_l(x)$ *there exists* $y' \in \Sigma'$ *such that* $|x'-y'| \le \alpha_1$, $d_\Sigma(x', y') \ge 2\alpha_1$ *and* $\langle n(x'), n(y')\rangle > 0$.

Proof. Fix $N \in \mathbb{N}$ such that $\frac{4l}{s} \le N < \frac{4l}{s} + 1$, s as in Corollary 4.6 and assume $\alpha_l < \min(\alpha_2, \frac{\alpha_2}{C_0^N})$, C_0 being the constant as in (2.4). Our goal is to find C_l, note that $\mathcal{B}_l(x) \subset \Sigma'$. Let $x' \in \mathcal{B}_l(x)$ then there exists a geodesic $\gamma(t)$, $t \in [0,1]$ such that $\gamma(0) = x$, $\gamma(1) = x'$ and length$(\gamma) \le l$. Fix a partition $\mathcal{Q}$ of $[0,1]$, $\mathcal{Q} = \{t_i \in [0,1] | 0 \le i \le T\}$, such that

$$\begin{cases} t_0 = 0, t_T = 1 \\ d_\Sigma(\gamma(t_i) = x_i, \gamma(t_{i+1}) = x_{i+1}) \le \frac{s}{4} \\ T \le N \end{cases} . \tag{4.10}$$

Since $|x-y| \le \alpha \le \alpha_2$ and $d_\Sigma(x,y) \ge C_l \ge 2\alpha_2$ then Lemma 4.5 gives $\mathcal{B}_s(x)$ and $\mathcal{B}_s(y)$ contain respectively a graph u_1 and u_2 over $\mathcal{D}_{\frac{s}{2}(x)}$ and $\sup_{\mathcal{D}_{\frac{s}{4}(x)}} |u_1 - u_2| \le C_0\alpha$. Therefore, let $z_1 \in \mathcal{D}_{\frac{s}{4}}(x)$ such that $x_1 = u_1(z_1)$ and let $y_1 = u_2(z_1)$, then

$$|x_1 - y_1| \le C_0\alpha \le \alpha_2,$$

$$d_\Sigma(x_1, y_1) \ge d_\Sigma(x,y) - d_\Sigma(x_1, x) - d_\Sigma(y_1, y) \ge C_l - \frac{s}{4} - \frac{5s}{4} \ge 2\alpha_2$$

if C_l is big enough, and $\langle N(x_1), N(y_1)\rangle > 0$. As long as $d_\Sigma(x_i, y_i) \ge 2\alpha_2$ we can apply Lemma 4.5. We can repeat this argument N times as long as $C_l - N\frac{3s}{2} \ge 2\alpha_2$. □

Corollary 4.8. *For each* $0 < l < \bar{l} - (1+\bar{\varepsilon})$ *there exist* $\alpha_l > 0$ *and* $C_l > 0$ *so: Given* $\alpha < \alpha_l$ *and* $x, y \in \Sigma^1$ *such that* $|x-y| \le \alpha$, $d_\Sigma(x,y) \ge C_l$ *and* $< n(x), n(y) >> 0$ *then for each* $x' \in \mathcal{B}_l(x)$, $\mathcal{B}_t(y)$ *contains a graph* $\{z + u(z)n(z)\}$ *over a domain containing* $\mathcal{B}_{\frac{t}{4}}(x)$, $|\nabla u| + |u| \le \alpha C_1$.

Proof. Apply Lemma 4.7 and Lemma 4.5. □

Corollary 4.9. *For each* $0 < l < \bar{l} - (1 + \bar{epsilon})$ *and* $\delta > 0$ *there exist* $\alpha_{l,\delta} > 0$ *and* $C_l > 0$ *so: Given* $\alpha < \alpha_{l,\delta}$ *and* $x, y \in \Sigma^1$ *such that* $|x - y| \leq \alpha$, $d_\Sigma(x, y) \geq C_l$ *and* $\langle N(x), N(y)\rangle > 0$ *then* $\mathcal{B}_l(x)$ *is* δ*-stable.*

Proof. Let $\alpha_{l,\delta} < \min(\alpha_l, \frac{\delta}{C_1})$, according to Corollary 2.5, we need to find a u which is a positive solution of the CMC graph equation over $\mathcal{B}_l(x)$. Corollary 4.8 gives that the latter is true locally. In fact, fix $x_i \in \mathcal{B}_l(x)$ such that $\mathcal{B}_s(x_i)$ and $\mathcal{B}_{\frac{s}{4}}(x_i)$ are both finite coverings for $\mathcal{B}_l(x)$. From Lemma 4.7 it follows that $\mathcal{B}_s(x_i)$ contains a graph $\{z + u_i(z)n(z)\}$ over a domain containing $\mathcal{B}_{\frac{s}{4}}(x_i)$, $|\nabla u_i| + |u_i| \leq \alpha C_1$. The function $u(y) := u_i(y)$ if $y \in \mathcal{B}_{\frac{s}{4}}(x_i)$ is a well defined function over $\mathcal{B}_l(x)$ such that $|\nabla u| + |u| \leq \alpha C_1 < \delta$. Applying Corollary 2.5 gives this corollary. □

4.2 The non-existence of large almost stable domain and the uniform bound

In this section we are going to need [21, **Theorem 0.1.**] by Sirong Zhang, Theorem 3.2, and the Bishop Volume Comparison Theorem [19, **Theorem 1.3.**]:

Theorem 4.10 (Bishop Volume Comparison Theorem). *Let* M *be an* n*-dimensional complete Riemannian manifold with* $Ric(M) \geq (n-1)K$*. Then for any* $x \in M$ *and* $R > 0$, $\frac{Vol(\mathrm{B}_R(x))}{V(K,R)}$ *is a non-increasing function in* R*. Hence,*

$$Vol(\mathrm{B}_R(x)) \leq V(K, R),$$

where $V(K, R)$ *is the volume of the geodesic ball of radius* R *in the space form* M_K*.*

Theorem 3.2 is essentially what determines how big $\bar{l}$ is. Our surface has trivial normal bundle since it is orientable and this will be proved later in Theorem 4.17. Since $\sup_{\Sigma \cap B_{\bar{l}}(0)} |A|^2 \leq 4C^2$ using the Gauss equation (2.2) gives a lower bound for the Gaussian curvature,

$$K_\Sigma \geq -4C^2 = 2G. \tag{4.11}$$

Using Theorem 4.10 this lower bound implies an upper bound on the area of the intrinsic balls. For any $x \in \Sigma$

$$Vol(B_R(x)) \leq V(G, R), \qquad G \text{ as in 4.11.} \tag{4.12}$$

The following proposition uses what we have proved in the previous sections and Theorem 3.2 to show that Σ' does not contain a large almost stable domain. Roughly speaking if we take l large and assume that $\mathcal{B}_l(x) \subset \Sigma^1$ is δ-stable, Theorem 3.2 implies that $\mathcal{B}_{\frac{l}{2}}(x)$ is almost flat. This forces the intrinsic disk to leave the unit ball.

Proposition 4.11. *Given $\delta > 0$ there exists $l_\delta > 0$ so: For any $l \geq l_\delta$ and $x \in \Sigma^1$ if $\mathcal{B}_l(x)$ is δ-stable then it is not contained in Σ^1.*

Proof. Let us fix $l_\delta > \max(\frac{20}{9}, \sqrt{\frac{C}{\rho}})$. Let C be as in Theorem 3.2 and ρ as in (4.5). Being $\mathcal{B}_l(x)$ δ-stable Theorem 3.2 implies

$$\sup_{\mathcal{B}_{\frac{l}{2}}(x)} |A|^2 \leq \frac{C}{l^2} \leq \rho.$$

(4.5) implies that

$$\inf_{\mathcal{B}_{\frac{l}{2}}(x)} \frac{|x-y|}{d_\Sigma(x,y)} \geq \frac{9}{10}.$$

Taking $y \in \mathcal{B}_{\frac{l}{2}}(x)$ such that $d_\Sigma(x,y) > \frac{10}{9}$, we have $|x-y| > 1$. This proves that $\mathcal{B}_l(x)$ is not contained in Σ^1. ☐

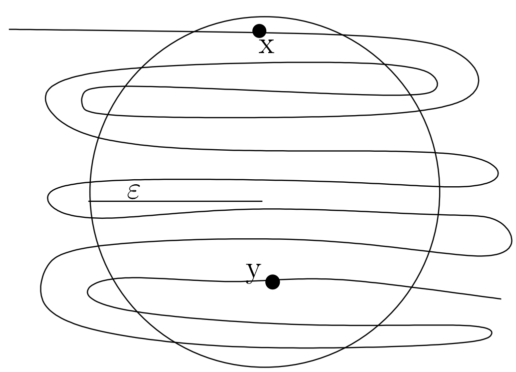

Figure 8: Intrinsically far apart

We have proved so far that decreasing the Euclidean distance between two points gives a large δ-stable domain as long as we increase their intrinsic distance. In the following lemma we apply the Bishop Volume Comparison Theorem and a lower bound on the area of each piece to prove that the more graphs there are in a small ball the larger the intrinsic distance becomes. This is what Fig. 8 illustrates.

Let us fix $\delta, \bar{\varepsilon} > 0$ small, let $l_1 = l_\delta$ as given by Proposition 4.11 and let $\bar{l}$ in Theorem 4.3 to be equal to $l_1 + 1 + \bar{\varepsilon}$. In other words,

Σ^1 does not contain a δ-stable geodesic ball of radius bigger than l_1.

Let us fix $0 < r < \alpha_{l_1,\delta}$ where $\alpha_{l_1,\delta}$ is taken as in Corollary 4.9. This means that

$$\text{For any } x \in \Sigma',\ B_r(x) \cap \Sigma' \text{ consists of graphs over } T_x\Sigma \tag{4.13}$$

and also that

$$\begin{aligned} &\text{There exists } C_{l_1} > 0 \text{ such that if } x, y \in B_r(x) \cap \Sigma^1,\ d_\Sigma(x,y) \geq C_{l_1} \\ &\text{and } \langle N(x), N(y) \rangle > 0 \text{ then } \mathcal{B}_{l_1}(x) \text{ is } \delta\text{-stable.} \end{aligned} \tag{4.14}$$

Given $x \in \Sigma^1$ let n_x be the number of components of $B_r(x) \cap \Sigma^1$. The area of each component Σ_i^x could go to zero if they accumulate toward the boundary of the ball. Nonetheless we have proven that these graphical pieces continue outside the ball, 4.7. Therefore we have a uniform lower bound on the area of Σ_i^x; in other words

$$\text{There exits } \varepsilon > 0 \text{ such that Area}(\Sigma_i^x) > \varepsilon \text{ independently of } x \text{ and } i. \tag{4.15}$$

Lemma 4.12. *Given $\lambda > 0$ there exists $n_\lambda > 0$ so: If $x \in \Sigma^1$ and $n_x > n_\lambda$ then there exist $y, y' \in \mathrm{B}_r \cap \Sigma^1$ such that $dist_\Sigma(y, y') > \lambda$ and $\langle n(y), n(y') \rangle > 0$.*

Proof. Theorem 4.10, that is, Bishop Volume Comparison Theorem, gives an upper bound for the area of $\mathcal{B}_\lambda(x)$, namely

$$\text{Area}(\mathcal{B}_\lambda(x)) < V(G, \lambda).$$

At this point it follows easily that if $n_x \in \mathbb{N}$, $n_x > \frac{V(G,\lambda)}{\varepsilon}$ then there exists $y_1 \in B_r \cap \Sigma^1$ which is not in $\mathcal{B}_\lambda(x)$, i.e. $\text{dist}_\Sigma(x, y_1) > \lambda$. $\langle n(x), n(y_1) \rangle > 0$ does not necessarily happen. Take $\bar{V} \in \mathbb{N}$ such that $\bar{V} - 1 < \frac{V(G,\lambda)}{\varepsilon} \leq \bar{V}$ and let $n_\lambda = \bar{V}^{\bar{V}}$. If $n_x \geq n_\lambda$ then there exist at least $\bar{V}$ distinct y_i, $i := 1, ...\bar{K}$, in different component of $B_r(x) \cap \Sigma^1$ such that $\text{dist}_\Sigma(x, y_i) > \lambda$. Fixed y_1 there exists y_2 among the y_i such that $\text{dist}_\Sigma(y_1, y_2) > \lambda$. At this point either $\langle n(x), n(y_1) \rangle > 0$ or $\langle n(x), n(y_2) \rangle > 0$ or $\langle n(y_1), n(y_2) \rangle > 0$. ☐

The following corollary uses Proposition 4.11 and Lemma 4.12 to obtain the upper bound on the number of graphs.

Corollary 4.13. *For any $x \in \Sigma^1$, $n_x \leq n_{C_{l_1}}$.*

Proof. If $n_x > n_{C_{l_1}}$ then Lemma 4.12 with λ equal to C_{l_1} gives that there exist $y, y' \in B_r(x) \cap \Sigma^1$ such that $\mathrm{dist}_\Sigma(y, y') > C_{l_1}$ and $\langle n(y), n(y') \rangle > 0$. Using Lemma 4.9 gives that $\mathcal{B}_{l_1}(x)$ is δ-stable and, by Proposition 4.11, cannot be contained in Σ^1. Since $\mathrm{dist}_\Sigma(y, y') > C_{l_1} > l_1$, y' is not in Σ^1. This gives the contradiction that implies $n_x \leq n_{C_{l_1}}$. □

4.3 Multi-valued graphs in CMC surfaces

The limit surface is embedded and minimal by a standard argument which will be sketched below. To prove that it is simply connected we need more work and well-known topological results.

Let $r > 0$ be as defined in (4.13) and (4.14), that is, there exists a finite covering $B_r(x_i^n)$ for Σ_n^1 where everything is graphical over $T_{x_i^n}\Sigma_n$ and the number of graphs is uniformly bounded. We can also assume that the number of balls involved is uniformly bounded with respect to n. Going to a subsequence, we can assume x_i^n converging to x_i and $T_{x_i^n}\Sigma_n$ converging to a certain $T_{x_i}\Sigma_\infty$. Using the argument outlined in Section **??**, the fact that the number of graphs is uniformly bounded and the maximum principle for minimal surfaces gives that the limit is an embedded minimal surface. Fig. 9 illustrates the two types of intersection that could occur if

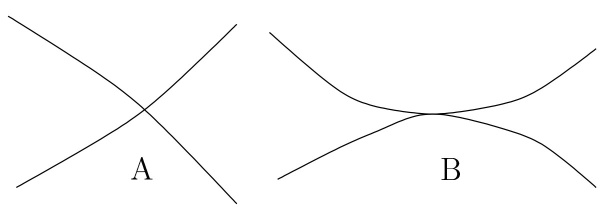

Figure 9: Embedded

the limit is not embedded: A cross intersection, type A, and a tangential intersection, type B. However type A cannot be a continuous limit of embedded surfaces. Type B, which could be the limit of a sequence of embedded surfaces, cannot occur because of the maximum principle for minimal surfaces. By continuity the curvature of this minimal surface is large at zero.

To prove that Σ_∞ is simply connected we use some results about Jordan curves [10] and the following theorems:

Theorem 4.14. *[10, **Chapter 3**] Every compact hypersurface in Euclidean space without boundary is orientable.*

Theorem 4.15. *[10, **Chapter 3**] If M is an orientable surface, so is M minus one point.*

Theorem 4.16. *[11, **Corollary 3.28.**] If M is a closed connected n-manifold, the torsion subgroup of $H_{n-1}(M, \mathbb{Z})$ is trivial if M is orientable and $\mathbb{Z}_2$ if M is not orientable.*

In our case Σ_∞ is a closed connected 2-manifold, and it is a consequence of Theorem 4.16 that, if it is orientable, the torsion subgroup of its fundamental group is trivial. The next proposition shows that Σ_∞ is the embedded minimal disk we have been looking for.

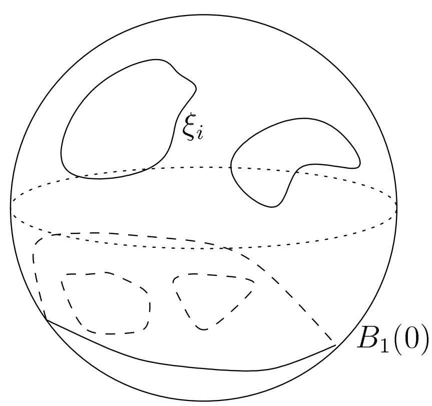

Figure 10: Boundary of Σ_∞

Proposition 4.17. *Σ_∞ is an embedded simply connected minimal surface such that, $0 \in \Sigma_\infty \subset B_1 \subset \mathbb{R}^3$, $\partial\Sigma_\infty \subset \partial B_1$, $\sup_{\Sigma\cap B_1} |A|^2 \leq 4C^2$ and $|A|^2(0) = C^2$.*

Proof. The conditions on the second fundamental form, that is

$$\sup_{\Sigma \cap B_1} |A|^2 \leq 4C^2 \text{ and } |A|^2(0) = C^2,$$

are a consequence of the C^2 convergence.

We have already proved that it is embedded and what we are left to prove is that Σ_∞ is simply connected. Let us prove that it is orientable first. We want to prove that Σ_∞ is homeomorphic to a compact embedded surface minus a finite number of points. Because it is an embedded minimal surface, $\partial\Sigma_\infty$ is a finite number of disjoint loops ξ_i, $i := 1, ..., I$ which do not have self intersections, Fig. 10. These loops lie on $\partial B_1(0)$ minus one point, hence we have essentially a finite number of Jordan curves in the plane. We can glue I disks to Σ_∞ in a way that the result is an embedded compact surface: Each Jordan curve ξ_i divides the plane into an inner and an outer region, and can be thought of as the boundary of a simply connected domain, namely a disk D_i. If a loop ξ_i lies in the inside of another loop ξ_j then we lift D_j so that it does not intersect D_i. Fig. 11 illustrates how we are gluing these disks to the surface. Since the number of loops is finite we repeat this a finite number of times and obtain

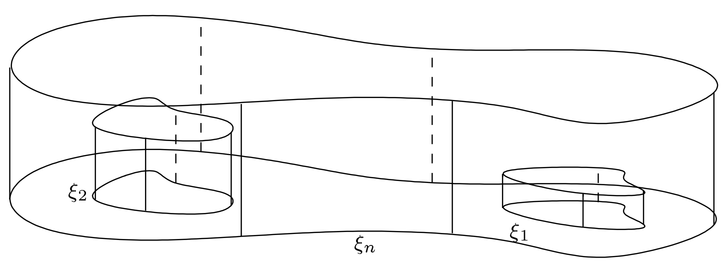

Figure 11: Orientable

in the end a new surface

$$\bar{\Sigma} = \{\Sigma_\infty \bigcup_{i=1}^{I} D_i\}/\sim$$

where $\sim$ is the relation that identify ξ_i with ∂D_i. $\bar{\Sigma}$ is a compact embedded surface without boundary and therefore orientable by Theorem 4.14. Theorem 4.15 implies that $\bar{\Sigma}$ take out a finite number of points is still orientable, that is $\bar{\Sigma} \setminus \bigcup_{i=1}^{I} D_i = \Sigma_\infty$. Theorem 4.16 tells that $\pi_1(\Sigma_\infty)$ is torsion free.

Let $\gamma : \ S^1 \longrightarrow \Sigma_\infty$ be a closed path and $B_\sigma(\gamma(t_i))$ a finite covering for γ such that $\sigma < \min(1-|\gamma|, r)$, and the number of components of $B_\sigma(\gamma(t))$ is non-increasing. This is possible after going to a subsequence, assuming n large because of the uniform bound. Fix a starting point $\gamma(t_0)$, an orientation on γ, and let

$$\gamma_n(t_0) := \gamma(t_0) + s_0 N_{\gamma(t_0)} \in \Sigma_n^1.$$

Moving continuously on $\gamma(t)$ we obtain a new path

$$\gamma_n(t) := \gamma(t) + s(t) N_{\gamma(t)} \in \Sigma_n^1.$$

The conditions on σ and n force the path to close up after it moves around γ a finite number of times, $k \in \mathbb{N}$. Since Σ_n^1 simply connected, there exists a map $\Gamma_n : \ D_1 \longrightarrow \Sigma_n^1$ such that $\Gamma_n|_{S^1} = \gamma_n$. Define $\Gamma : \ D_1 \longrightarrow \Sigma_\infty$,

$$\Gamma(x) := \lim_{n \longrightarrow \infty} \Gamma_n(x).$$

The existence of this map Γ proves that $k\gamma$ is homotopic to a point. Since $\pi_1(\Sigma_\infty)$ is torsion free, this implies that γ itself is homotopic to a point. Since γ could be any path on Σ_∞ we have proved that Σ_∞ is simply connected. $\square$

Finally we prove that Σ_n^1 and therefore Σ_n contains a multi-valued graph. In Proposition 4.17 we proved that Σ_∞ is an embedded simply connected minimal surface such that,

$$0 \in \Sigma_\infty \subset B_1 \subset \mathbb{R}^3, \partial\Sigma_\infty \subset \partial B_1, \sup_{\Sigma \cap B_1} |A|^2 \leq 4C^2 = 4|A|^2(0).$$

Taking $C = C(N, \omega, \varepsilon)$ as in Theorem 4.2, the same theorem gives that Σ_∞ contains and N-valued graph. Let u be this N-valued graph, defined over $\{(\rho, \theta) | r \leq \rho \leq s, |\theta| \leq N\pi\}$ as described in Definition 4.1. This is how we build an N-valued graph in Σ_n: Given $r \leq \bar{\rho} \leq s$, define $u_{\bar{\rho}}(\theta) = u(\bar{\rho}, \theta)$. Consider $u_{\bar{\rho}}$ as a path on Σ_∞ starting at $u_{\bar{\rho}}(-N\pi)$. Assuming n large, Σ_n^1 moves closer and closer to Σ_∞ and there exists a continuous function $\phi(\theta)$ such that

$$u_{\bar{\rho}}^n(\theta) = u_{\bar{\rho}}(\theta) + \phi(\theta)\mathbf{e}_3 \in \Sigma_n^1$$

is well defined. The function $u^n(\rho,\theta) = u^n_\rho(\theta)$ defined over $\{(\rho,\theta)|r \le \rho \le s, |\theta| \le N\pi\}$ is an N-valued graph.

Notice that as Σ^1_n moves closer and closer to Σ_∞, Σ^1_n and Σ_∞ are "parallel surfaces." Not only does Σ_n contain an N-valued graph, but the properties of this graph, such as the upper bound on the gradient, are preserved.

5 Example

In this chapter we provide examples of CMC surfaces containing arbitrary large multi-valued graphs. We use the method of successive approximations to build a sequence of normal variations of the helicoid that converges to an embedded and simply connected CMC surface containing a multi-valued graph.

5.1 Preliminaries

Let $\Sigma_h = \{x + u(x)N_\Sigma(x), x \in \Sigma\}$ be a normal variation of Σ, where Σ is any minimal surface. Σ_h is a CMC surface with mean curvature equal to H if $u(x)$ satisfies the following equation [12, 13, 15]:

$$Lu = H + Q(u), \qquad \text{where } Lu = \Delta u + |A|^2 u \tag{5.1}$$

is the linearized operator. Q is a quadratic and higher order function in u, u_i, u_{ij} where $i,\ j \in \{1, 2\}$, with geometric invariants of Σ as coefficients. Before we prove the existence of a constant mean curvature normal variation of the helicoid we need to describe some properties of the function Q.

Let $C^{k,\lambda}(\Sigma)$ be the standard subset of $C^k(\Sigma)$ consisting of functions whose k-th partial derivatives are Hölder continuous with exponent λ in Σ and let $\|\cdot\|_{k,\lambda}$ be the notation for the Hölder norm. Let us define Ω_δ, subset of $C^{2,\lambda}(\Sigma)$, in the following way:

$$\Omega_\delta := \{u \in C^{2,\lambda} : \|u\|_{2,\lambda} < \delta\}. \tag{5.2}$$

[12, Lemma C.2] gives that Q is a quadratic and higher order function. As a consequence, we have the following lemma, that says that $\|Q(u)\|_{0,\lambda}$ decays faster than $\|u\|_2\|u\|_{2,\lambda}$.

Lemma 5.1. *There exist $\delta_1 > 0$ and $C_1 > 0$ so: If $u \in C^{2,\lambda}(\Sigma)$ and $|A||u|, |u_i| < \delta_1$ then*

$$\|Q(u)\|_{0,\lambda} < C_1\|u\|_2\|u\|_{2,\lambda}.$$

As a consequence of Lemma 5.1, we have a new corollary that relates $\|Q(u)\|_{0,\lambda}$ and $\|u\|_{2,\lambda}$:

Corollary 5.2. *Given* $1 > C_2 > 0$ *there exists* $\delta_2 > 0$ *so: If* $u \in \Omega_{\delta_2}$ *and* $|A||u| < \delta_2$ *then*

$$\|Q(u)\|_{0,\lambda} < C_2\|u\|_{2,\lambda}.$$

Proof. Let $\delta_2 < \min(\frac{C_2}{C_1}, \delta_1)$. This implies that $C_1\|u\|_2 < C_1\|u\|_{2,\lambda} < C_1\delta_2 < C_2$ and that $|A||u|, |u_i| < \delta_2 < \delta_1$. Therefore, we can apply 5.1 and we have

$$\|Q(u)\|_{0,\lambda} < C_1\|u\|_2\|u\|_{2,\lambda} < C_2\|u\|_{2,\lambda}.$$

□

5.2 Normal Variation of the Helicoid

Let Σ be a simply connected disk in the helicoid that contains a multi-valued graph. Due to the domain monotonicity and continuity of eigenvalues [1] we can also assume that 0 is not an eigenvalue for L on Σ and therefore that the Dirichlet

$$\begin{cases} Lu = w \\ u|_{\partial\Sigma} = 0 \end{cases} \tag{5.3}$$

has a unique solution $u \in C^{2,\lambda}(\Sigma)$ for any $w \in C^{0,\lambda}(\Sigma)$. Assuming that 0 is not an eigenvalue for L gives also the following lemma [9, Theorem 5.3 and page 109]:

Lemma 5.3. *There exists a constant* B *depending only on* Σ *so: Let* $u \in C^{2,\lambda}(\Sigma)$ *be the unique solution for* (5.3) *then*

$$\|u\|_{2,\lambda} < B\|w\|_{0,\lambda}. \tag{5.4}$$

We will prove that there exists $H > 0$ such that a solution u for the Dirichlet problem

$$\begin{cases} Lu = H + Q(u) \\ u|_{\partial\Sigma} = 0 \end{cases} \tag{5.5}$$

exists and $\|u\|_{L^\infty}$ is small. The existence of a fixed neighborhood of the helicoid where the normal exponential map is injective guarantees that $\Sigma_H = \{x+u(x)N_\Sigma(x), x \in \Sigma\}$ is embedded, if $\|u\|_{L^\infty}$ is small enough. What we are about to show is that if H is small enough we can build a sequence of normal variations u^n of the helicoid that converges to a CMC normal variation u. We will also show that $\|u\|_{L^\infty}$ can be as small as we want and consequently that the CMC normal variation is embedded.

Let u^1 be the unique solution for

$$\begin{cases} Lu^1 = H \\ u^1|_{\partial\Sigma} = 0 \end{cases} \tag{5.6}$$

and u^n be the unique solution for

$$\begin{cases} Lu^n = H + Q(u^{n-1}) \\ u^n|_{\partial\Sigma} = 0 \end{cases} . \tag{5.7}$$

Lemma 5.3 implies that

$$\|u^1\|_{2,\lambda} < BH$$

and also that

$$\|u^k\|_{2,\lambda} < B(H + \|Q(u^{k-1})\|_{0,\lambda}). \tag{5.8}$$

The existence of a solution for

$$Lu = H + Q(u)$$

will follow clearly, and we will see how, applying Arzela-Ascoli to the sequence u^n if we prove that there exists a constant K such that $\|u^n\|_{2,\lambda} < K$ uniformly in n. Fix C_2 in Corollary 5.2 so that $\varepsilon = C_2 B < 1$, B as in Lemma 5.3. We will prove by strong induction that

if H is so that $BH(1 + \frac{1}{1-\varepsilon}) < \delta_2$ then $u^n \in \Omega_{\delta_2}$ for any n,

that is what we wanted. We have already that

$$\|u^1\|_{2,\lambda} < BH < \delta_2,$$

namely the statement is true for $n = 1$. Let us prove that

"true for $n = 1$ implies true for $n = 2$."

Assuming $\|u^1\|_{2,\lambda} < \delta_2$, we can apply Lemma 5.1 that gives that $\|Q(u^1)\|_{0,\lambda} < C_2\|u^1\|_{2,\lambda} < \varepsilon H$, therefore

$$\|u^2\|_{2,\lambda} \leq B(H + \|Q(u^1)\|_{0,\lambda} \leq B(H + \varepsilon H) < \delta_2. \tag{5.9}$$

Let us prove that

"true for all k with $k \leq n$ implies true for $k = n + 1$."

"True for all k with $k \leq n$" means that $\|u^k\|_{2,\lambda} < \delta_2$ for $k \leq n$ and therefore Lemma 5.1 gives that

$$\|Q(u^k)\|_{0,\lambda} < C_2\|u^k\|_{2,\lambda} < C_2 B(H + \|Q(u^{k-1})\|_{0,\lambda}) < \varepsilon(H + \|Q(u^{k-1})\|_{0,\lambda}) \tag{5.10}$$

for $k \leq n$. Applying (5.10) n times we have

$$\begin{aligned}\|u^{n+1}\|_{2,\lambda} &< B(H + \|Q(u^n)\|_{0,\lambda}) < B(H + \varepsilon(H + \|Q(u^{n-1})\|_{0,\lambda})) < \\ &< B(H + \varepsilon(H + \varepsilon(H + \|Q(u^{n-2})\|_{0,\lambda}))) < \\ &< BH(1 + \sum_{k=1}^{n} \varepsilon^k) < BH(1 + \frac{1}{1-\varepsilon}) < \delta_2. \end{aligned} \tag{5.11}$$

Now that we have proved that $\|u^n\|_{2,\lambda} < \delta_2$ uniformly in n, using Arzela-Ascoli we can extract a subsequence that converges C^2 to a certain $u \in C^2(\Sigma)$. Taking the limit as n goes to infinity on both sides of the equation

$$Lu^n = H + Q(u^{n-1})$$

gives that

$$Lu = H + Q(u).$$

u is therefore a constant mean curvature normal variation of the helicoid. It is clear from the proof that taking H small gives $\|u\|_{L^\infty}$ small. Consequently, the constant mean curvature normal variation that we have built is also embedded.

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