On Convergence of Singular Series for a Pair of Quadratic Forms

by

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Abstract

Examining the system of Diophantine equations

$$\begin{cases} f_1(x) = x_1^2 + ...x_n^2 = \nu_1, \\ f_2(x) = \lambda_1 x_1^2 + ...\lambda_n x_n^2 = \nu_2, \end{cases}$$

with $\lambda_i \neq \lambda_j$ and $\nu_i, \lambda_i \in \mathbb{Z}$, we show that the singular series $S(\nu)$ converges if $n \geq 6$.

READERS: Takashi Ono (Advisor).

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Contents

Chapter 1

Introduction

1.1 The Classical Waring's Problem

The story begins almost ninety years ago with Hardy and Littlewood, who examined the integer solutions for the equation

$$x_1^\delta + x_2^\delta + ... + x_n^\delta = \nu, \quad \nu \in \mathbb{Z}$$

for an integer $\delta \geq 2$. To put this another way, if f is a polynomial function such that $f : \mathbb{Q}^n \to \mathbb{Q}$ where $\nu \in \mathbb{Q}$, we wish to study the set $f^{-1}(\nu)$ (i.e. the fiber of f which maps down to ν). Classically, to describe this set, one would first define the ring of (rational) adeles as follows. Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the p-adic norm, and let $\mathbb{Q}_\infty = \mathbb{R}$ (i.e. the completion of $\mathbb{Q}$ with respect to the usual absolute value). Then for any finite set of primes S (possibly including ∞), we can define

$$\mathbb{Q}_{\mathbb{A}}(S) = \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p.$$

The union of these $\mathbb{Q}_{\mathbb{A}}(S)$ over all possible S is called the *ring of adeles*, denoted $\mathbb{Q}_{\mathbb{A}}$.

Using this, we can define

$$N_t(\nu) = \#(f^{-1}(\nu) \bigcap K_t) = \#\{x \in K_t : f(x) = \nu\} = \sum_{x \in K_t : f(x) = \nu} 1$$

such that

$$\bigcup_t K_t = \mathbb{Q}_{\mathbb{A}}^n,$$

where the $\{K_t\}$ is a family of compact sets in $\mathbb{Q}_{\mathbb{A}}^n$ which can be parameterized by t in such a way that $\lim_{t\to\infty} K_t = \mathbb{Q}_{\mathbb{A}}^n$. One would then consider

$$\lim_{t\to\infty} \frac{N_t(\nu)}{t^*},$$

where t^*, loosely speaking, is t to some non-negative power. This type of quantity is known as a *singular series*, denoted $\mathfrak{S}(\nu)$. Hardy and Littlewood made extensive use of the singular series; using a method which they called the circle method, they showed that the above quantity grew sufficiently quickly as $t \to \infty$ to ensure that $N(\nu)$ itself was guaranteed to be non-zero for sufficiently large ν.

In this paper, while preserving the spirit of the Hardy-Littlewood computation, we take a slightly different tack in devising our version of the singular series. This allows us to apply the ideas of Hardy and Littlewood to larger classes of functions, and it allows us to ask a broader range of questions about these functions. We describe these methods and the function of particular interest to this paper in the next section.

1.2 New Formulations for $N(t)$ and $S(t)$

We state here the new ideas for N and S in more generality than is necessary for the specific example discussed in this paper. Let k be an $\mathbb{A}$-field, let $k_{\mathbb{A}}$ be the ring of adeles over k, and let S^1 denote the group of complex numbers of absolute value 1. We fix a basic character χ of k, i.e. let $\chi : k_{\mathbb{A}} \to S^1$ be a homomorphism such that $\chi(k) = 1$ but χ is non-trivial. Let $f : k^n \to k^m$ be a polynomial map over k. We use the same notation f for maps obtained from f over k_v, $k_{\mathbb{A}}$ in the natural way.

Next, we consider the function

$$\varphi : k_{\mathbb{A}}^n \to \mathbb{C}$$

and use it to define

$$N_\varphi(\nu) = \sum_{\gamma \in k^n, f(\gamma)=\nu} \varphi(\gamma). \qquad (*)$$

This sum can be thought of as $N_{f,\varphi}(\nu)$. We assume that φ is a function in what is known as the Schwartz space, denoted $\mathcal{S}(k_{\mathbb{A}}^n)$. The assumption of $\varphi \in \mathcal{S}(k_{\mathbb{A}}^n)$ implies the following properties:

$$\sum_{\gamma \in k^n} |\varphi(\gamma)| < +\infty, \text{ i.e. } \varphi|_{k^n} \in L^1(k^n). \qquad (A.1)$$

$$\varphi \in L^1(k_{\mathbb{A}}^n). \qquad (A.1)_{\mathbb{A}}$$

Let

$$(\Gamma\varphi)(\xi) = \sum_{\gamma \in k^n} \varphi(\gamma)\chi(< f(\gamma), \xi >). \qquad (**)$$

This is clearly integrable, since $|\chi(< f(\gamma), \xi >)| = 1$. Moreover, because $f(\gamma) \in k^m$, it is clear that $\Gamma\varphi$ is a function modulo k^m, i.e.

$$(\Gamma\varphi)(\xi + a) = (\Gamma\varphi)(\xi), \ \forall \ a \in k^m$$

Also , for $\nu \in k^m$, the "ν-th Fourier coefficient of $(\Gamma\varphi)(\xi)$" is

$$\int_{k_{\mathbb{A}}^m/k^m} (\Gamma\varphi)(\xi)\bar{\chi}(< \xi, \nu >)d\xi$$
$$= \sum_{\gamma \in k^n} \varphi(\gamma) \int_{k_{\mathbb{A}}^m/k^m} \chi(< f(\gamma) - \nu, \xi >)d\xi$$

Let

$$\psi(\xi) = \chi(< f(\gamma) - \nu, \xi >).$$

Note that $\psi \in \mathrm{Hom}(k_{\mathbb{A}}^m/k^m, \mathbb{S}^1)$. Hence we have

$$\int_{k_{\mathbb{A}}^m/k^m} \psi(\xi) d\xi = \begin{cases} 1 & \text{if } \psi = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So this integral is 1 if and only if $f(\gamma) = \nu$. Thus,

$$\begin{aligned} &\sum_{\gamma \in k^n} \varphi(\gamma) \int_{k_{\mathbb{A}}^m/k^m} \chi(< f(\gamma) - \nu, \xi >) d\xi \\ &= \sum_{\gamma \in k^n, f(\gamma) = \nu} \varphi(\gamma) \\ &= N(\nu). \end{aligned}$$

This means that $(*)$ is the "ν-th Fourier coefficient of $(**)$".

Now, we can move from the sum over k^n to the integral over $k_{\mathbb{A}}^n$, because of adelized axiom $(\mathrm{A.1})_{\mathbb{A}}$. For $\xi \in k_{\mathbb{A}}^m$, we let

$$\mathcal{G}\varphi(\xi) = \int_{k_{\mathbb{A}}^n} \varphi(x) \chi(< f(x), \xi >) dx.$$

By $(\mathrm{A.1})_{\mathbb{A}}$, this integral converges. We let

$$S(\nu) = \int_{k_{\mathbb{A}}^m} \mathcal{G}\varphi(\xi) \bar{\chi}(< \xi, \nu >) d\xi.$$

Clearly, $S(\nu) = \widehat{\mathcal{G}\varphi}(\nu)$, the Fourier transform of $\mathcal{G}\varphi$. In order for this to be useful, we require an implicit condition:

$$\mathcal{G}\varphi \in L^1(k_{\mathbb{A}}^m). \tag{A.2}$$

This allows the integral $S(\nu)$ to converge.

1.3 The Intersection of Two Quadratics

This new formulation of the singular series allows us to ask the same Waring-type questions about other polynomials or m-tuples of polynomials as well. In this section, we will consider the specific case of

$$K = \mathbb{Q},$$

$$f : \mathbb{Q}^n \to \mathbb{Q}^2,$$

$$x = (x_1, x_2, ..., x_n),$$

$$f(x) = (f_1(x), f_2(x)),$$

where n is even and

$$f_1(x) = x_1^2 + ...x_n^2,$$

$$f_2(x) = \lambda_1 x_1^2 + ...\lambda_n x_n^2,$$

We will assume $\lambda_i \neq \lambda_j \ \forall \ i \neq j$. For future ease, we note that $f_1(x) = |x|^2$.

Of course, we are interested in whether a pair of natural numbers can be represented by such a pair of functions; i.e., for $\nu = (\nu_1, \nu_2) \in \mathbb{N}^2$, we examine when

$$f_1(x) = \nu_1,$$

$$f_2(x) = \nu_2.$$

We require a Schwartz function φ which allows the axioms in the previous section to be satisfied. We will take

$$\varphi(x) = \prod_v \varphi_v(x) = \varphi_\infty(x) \prod_p \varphi_p(x),$$

where

$$\varphi_p(x) = \text{the characteristic function on } \mathbb{Z}_p,$$

$$\varphi_\infty(x) = e^{-\pi|x|^2}.$$

Since $\bigcap_p \mathbb{Z}_p = \mathbb{Z}$, we note that $\prod_p \varphi_p(x)$ is the characteristic function on $\mathbb{Z}$. So

$$\begin{aligned} N_\varphi(\nu) &= \sum_{\gamma \in k^n, f(\gamma)=\nu} \varphi(\gamma) \\ &= \sum_{\gamma \in \mathbb{Z}^n, f(\gamma)=\nu} e^{-\pi f_1(\gamma)} \\ &= e^{-\pi \nu_1} \sum_{\gamma \in \mathbb{Z}^n, f(\gamma)=\nu} 1. \end{aligned}$$

So $e^{\pi\nu_1} N(\nu)$ gives the exact number of solutions to the equation $f(x) = \nu$.

Now, we approximate $N(\nu)$ with our singular series $S(\nu)$. First, for $x_p \in \mathbb{Q}_p$, we let $\{x_p\}$ denote the fractional part of x_p; i.e.

$$\{x_p\} \equiv x_p \pmod{\mathbb{Z}_p}.$$

We let χ be a basic character of $\mathbb{Q}$; in particular, we will let

$$\chi(x) = \prod_v \chi_v(x_v)$$

be the product over the various valuations of $\mathbb{Q}$, where

$$\chi_v(x_v) = \begin{cases} e^{-2\pi i x_\infty} & \text{if } v = \infty, \\ e^{2\pi i \{x_p\}} & \text{if } v \neq \infty,\ v = p. \end{cases}$$

In this case, given our definitions above, we have

$$\mathcal{G}\varphi(\xi) = \int_{\mathbb{Q}_\mathbb{A}^n} \varphi(x)\chi(< f(x), \xi >)dx,$$

which means that

$$S(\nu) = \int_{\mathbb{Q}_\mathbb{A}^2} \mathcal{G}\varphi(\xi)\bar{\chi}(< \xi, \nu >)d\xi.$$

In the upcoming parts of this section, we will consider the localizations of this function for the various places of $\mathbb{Q}$. Such a localization is

$$S_p(\nu) = \int_{\mathbb{Q}_v^2} \mathcal{G}\varphi(\xi)\bar{\chi}(< \xi, \nu >)d\xi.$$

We will consider separately the places where v is infinite and $v = p$ is finite.

Chapter 2

$v = \infty$: $f = (f_1, f_2)$ over $\mathbb{R}$

2.1 Convergence

First, we consider the case where $v = \infty$, which means that $\mathbb{Q}_v = \mathbb{R}$. Before evaluating $S_p(\nu)$, we must show that $|\mathcal{G}_f\varphi(\xi)| \in L^1(\mathbb{R}^2)$, i.e.

$$\int_{\mathbb{R}^2} |\mathcal{G}_f\varphi_\infty(\xi)| d\xi < \infty.$$

This will allow us to use Fubini's Theorem.

Throughout this section, we will use $\mathcal{G}_f\varphi$ and $\mathcal{G}_f\varphi_\infty$ interchangeably when the context is clear. By our definitions,

$$\begin{aligned}
\mathcal{G}_f\varphi(\xi) &= \int_{\mathbb{R}^n} e^{-\pi|x|^2} \chi(f_1(x)\xi_1 + f_2(x)\xi_2) dx \\
&= \int_{\mathbb{R}^n} e^{-\pi \sum_{i=1}^n x_i^2} e^{-2\pi i (\sum_{i=1}^n x_i^2 \xi_1 + \sum_{i=1}^n \lambda_i x_i^2 \xi_2)} dx \\
&= \int_{\mathbb{R}^n} e^{-\pi \sum_{i=1}^n x_i^2 (1 + 2i(\xi_1 + \lambda_i \xi_2))} dx \\
&= \prod_{i=1}^n \int_{\mathbb{R}} e^{-\pi x_i^2 (1 + 2i(\xi_1 + \lambda_i \xi_2))} dx.
\end{aligned}$$

So

$$\int_{\mathbb{R}^2} |\mathcal{G}_f\varphi(\xi)| d\xi = \int_{\mathbb{R}^2} \prod_{i=1}^n \left| \int_{\mathbb{R}} e^{-\pi x_i^2 (1 + 2i(\xi_1 + \lambda_i \xi_2))} dx \right| d\xi.$$

By a computation of T. Ono, we know that

$$\int_{\mathbb{R}} e^{-\pi x_i^2(1+2i(\xi_1+\lambda_i\xi_2))}dx = e^{-\frac{in}{2}\tan^{-1}(2(\xi_1+\lambda_i\xi_2))}\frac{1}{(1+4(\xi_1+\lambda_i\xi_2)^2)^{\frac{1}{4}}}.$$

So the above integral is

$$\int_{\mathbb{R}^2}\prod_{i=1}^{n}\frac{1}{(1+4(\xi_1+\lambda_i\xi_2)^2)^{\frac{1}{4}}}d\xi$$
$$=\int_{\mathbb{R}^2}\prod_{i=1}^{\frac{n}{2}}\frac{1}{(1+4(\xi_1+\lambda_i\xi_2)^2)^{\frac{1}{4}}}\frac{1}{(1+4(\xi_1+\lambda_{i+\frac{n}{2}}\xi_2)^2)^{\frac{1}{4}}}d\xi.$$

Now, let us review Hölder's inequality. As always, we let $||\cdot||_p$ denote the L_p-norm. The basic form of Hölder's is the following statement:

Hölder's Inequality. *Let $f \in L_p$ and $g \in L_q$, where $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$||fg||_1 \leq ||f||_p||g||_q.$$

Now, instead of using Hölder's for two equations f and g, we require a generalization to r equations $f_1, ..., f_r$:

Generalized Hölder's Inequality. *Let $f_1, ...f_r$ be such that $f_i \in L_{p_i}\ \forall\ i$, where $\sum_{i=1}^{r}\frac{1}{p_i}=1$. Then*

$$||\prod_{i=1}^{r} f_i||_1 \leq \prod_{i=1}^{r}||f_i||_{p_i}.$$

The proof is a simple exercise in induction.

Now, we can apply this in our case by letting $r=\frac{n}{2}$, and

$$f_i = \frac{1}{(1+4(\xi_1+\lambda_i\xi_2)^2)^{\frac{1}{4}}}\frac{1}{(1+4(\xi_1+\lambda_{i+\frac{n}{2}}\xi_2)^2)^{\frac{1}{4}}}.$$

Since the f_i are all positive, it is clear that

$$||\prod_{i=1}^{\frac{n}{2}} f_i||_1 = \int_{\mathbb{R}^2}\prod_{i=1}^{\frac{n}{2}} f_i d\xi.$$

So

$$|\mathcal{G}_f\varphi(\xi)| = \int_{\mathbb{R}^2} \prod_{i=1}^{\frac{n}{2}} f_i d\xi \leq \prod_{i=1}^{\frac{n}{2}} ||f_i||_{\frac{n}{2}} = \prod_{i=1}^{\frac{n}{2}} (\int_{\mathbb{R}^2} (f_i)^{\frac{n}{2}} d\xi)^{\frac{2}{n}},$$

or, equivalently,

$$|\mathcal{G}_f\varphi(\xi)| \leq \prod_{i=1}^{\frac{n}{2}} (\int_{\mathbb{R}^2} (\frac{1}{(1+4(\xi_1+\lambda_i\xi_2)^2)^{\frac{1}{4}}} \frac{1}{(1+4(\xi_1+\lambda_{i+\frac{n}{2}}\xi_2)^2)^{\frac{1}{4}}})^{\frac{n}{2}} d\xi)^{\frac{2}{n}}$$
$$= \prod_{i=1}^{\frac{n}{2}} (\int_{\mathbb{R}^2} (\frac{1}{(1+4(\xi_1+\lambda_i\xi_2)^2)})^{\frac{n}{8}} (\frac{1}{(1+4(\xi_1+\lambda_{i+\frac{n}{2}}\xi_2)^2)})^{\frac{n}{8}} d\xi)^{\frac{2}{n}}.$$

Let us change variables to let

$$z = \xi_1 + \lambda_{i+\frac{n}{2}}\xi_2,$$
$$w = \xi_1 + \lambda_i\xi_2.$$

Let J_i be the Jacobian of the transformation above. It is easy to see that

$$J_i = \frac{1}{\lambda_i - \lambda_{i+\frac{n}{2}}}.$$

By our assumption that the λ's are unequal, this change of variables is non-degenerate.

Moreover, let

$$J = \prod_{i=1}^{\frac{n}{2}} J_i.$$

Then the above integral is

$$\prod_{i=1}^{\frac{n}{2}} (\int_{\mathbb{R}^2} J_i (\frac{1}{(1+4(w)^2)})^{\frac{n}{8}} (\frac{1}{(1+4(z)^2)})^{\frac{n}{8}} dzdw)^{\frac{2}{n}}$$
$$= J^{\frac{2}{n}} \prod_{i=1}^{\frac{n}{2}} (\int_{\mathbb{R}^2} (\frac{1}{(1+4(w)^2)})^{\frac{n}{8}} (\frac{1}{1+4(z)^2})^{\frac{n}{8}} dzdw)^{\frac{2}{n}}$$
$$= J^{\frac{2}{n}} \prod_{i=1}^{\frac{n}{2}} (\int_{\mathbb{R}} \left(\frac{1}{1+4(w)^2}\right)^{\frac{n}{8}} dw \int_{\mathbb{R}} (\frac{1}{1+4(z)^2})^{\frac{n}{8}} dz)^{\frac{2}{n}}$$
$$= J^{\frac{2}{n}} \prod_{i=1}^{\frac{n}{2}} (\int_{\mathbb{R}} (\frac{1}{1+4(w)^2})^{\frac{n}{8}} dw)^{\frac{4}{n}}.$$

But we know the integral

$$\int_{\mathbb{R}} (\frac{1}{(1+4(w)^2)})^{\frac{n}{8}} dw$$

converges if $n \geq 6$ and diverges if $n \leq 4$. Thus, if $n \geq 6$ and n is even then $\mathcal{G}\varphi(\xi) \in L^1(\mathbb{R}^2)$.

As a corollary, if n is odd and $n \geq 6$ then $\mathcal{G}\varphi \in L^1(\mathbb{R}^2)$. This follows from the fact that, if n odd,

$$\int_{\mathbb{R}^2} \prod_{i=1}^{n} \frac{1}{(1+4(\xi_1+\lambda_i\xi_2)^2)^{\frac{1}{4}}} d\xi \leq \int_{\mathbb{R}^2} \prod_{i=1}^{n-1} \frac{1}{(1+4(\xi_1+\lambda_i\xi_2)^2)^{\frac{1}{4}}} d\xi,$$

since

$$\frac{1}{(1+4(\xi_1+\lambda_n\xi_2)^2)^{\frac{1}{4}}} \leq 1 \ \forall \ \xi.$$

Hence if the integral converges for $n-1$ then it converges for n, and we know that $n-1 \geq 6$.

2.2 Evaluation of $S(\nu)$

Naturally, throughout this section we will assume $n \geq 6$. The goal of this section is to prove that, in the case described previously, we have the following:

Theorem 2. *Let*

$$K = \frac{1}{2} e^{-\pi(\nu_1)} (\nu_1)^{\frac{n-2}{2}}.$$

Additionally, let J_0 denote the 0-th Bessel function of second type, and let rect denote the usual rectangle function; both of these terms will be defined more explicitly later in the chapter. Moreover, let u_i vary over the unit ball, and let dV be the usual measure of this ball in hyperspherical coordinates. Then

$$\frac{S(\nu)}{K} = \pi \int_{S^{n-2}} \frac{2 \cdot \mathrm{rect}(\frac{1}{2} + \frac{(\lambda_n - \frac{\nu_2}{\nu_1})}{\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)u_i^2})}{|\pi(\nu_1)\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)u_i^2| \sqrt{1 - (1 + \frac{2(\lambda_n - \frac{\nu_2}{\nu_1})}{\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)u_i^2})^2}} dV.$$

Proof. To begin the proof of this theorem, we write the definition of $S(\nu)$:

$$\begin{aligned}
S(\nu) &= \int_{\mathbb{R}^2} G_f\varphi(\xi)\bar{\chi}(<\xi,\nu>)d\xi \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^n} \varphi(x)\chi(<f(x),\xi>)\bar{\chi}(<\xi,\nu>)dxd\xi \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^n} e^{-\pi|x|^2}e^{-2\pi i((\sum_{i=1}^n x_i^2)\xi_1+(\sum_{i=1}^n \lambda_i x_i^2)\xi_2)}e^{2\pi i<\xi,\nu>}dxd\xi \\
&= \int_{\mathbb{R}^n} e^{-\pi|x|^2}\int_{\mathbb{R}^2} e^{-2\pi i((\sum_{i=1}^n x_i^2)\xi_1+(\sum_{i=1}^n \lambda_i x_i^2)\xi_2)}e^{2\pi i(\xi_1\nu_1+\xi_2\nu_2)}d\xi dx.
\end{aligned}$$

Now, the first step will be to get rid of one of the integrals, specifically the one with respect to ξ_1. To facilitate this process, let us change variables so as to eliminate ξ_1 by using

$$z = \xi_1\nu_1 + \xi_2\nu_2,$$

i.e.

$$\begin{aligned}
\xi_1 &= \frac{z-\xi_2\nu_2}{\nu_1}, \\
d\xi_1 &= \frac{1}{\nu_1}dz.
\end{aligned}$$

Then

$$\begin{aligned}
S(\nu) &= \frac{1}{\nu_1}\int_{\mathbb{R}^n} e^{-\pi|x|^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i((\sum_{i=1}^n x_i^2)(\frac{z}{\nu_1}-\frac{\xi_2\nu_2}{\nu_1})+(\sum_{i=1}^n \lambda_i x_i^2)\xi_2)}e^{2\pi i(z)}dzd\xi_2dx \\
&= \frac{1}{\nu_1}\int_{\mathbb{R}^n} e^{-\pi|x|^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i((\sum_{i=1}^n x_i^2)\frac{z}{\nu_1}+(\sum_{i=1}^n (\lambda_i-\frac{\nu_2}{\nu_1})x_i^2)\xi_2)}e^{2\pi iz}dzd\xi_2dx \\
&= \frac{1}{\nu_1}\int_{\mathbb{R}^n} e^{-\pi|x|^2}\int_{\mathbb{R}} e^{-2\pi i(\sum_{i=1}^n (\lambda_i-\frac{\nu_2}{\nu_1})x_i^2)\xi_2}d\xi_2\int_{\mathbb{R}} e^{-2\pi iz(|x|^2\frac{1}{\nu_1}-1)}dzdx.
\end{aligned}$$

Now, we will use the fact that the Fourier transform of $e^{-\pi X^2}$ is itself. Define I_ϵ by

$$I_\epsilon = \frac{1}{\nu_1}\int_{\mathbb{R}^n} e^{-\pi|x|^2}\int_{\mathbb{R}} e^{-2\pi i(\sum_{i=1}^n (\lambda_i-\frac{\nu_2}{\nu_1})x_i^2)\xi_2}d\xi_2\int_{\mathbb{R}} e^{-2\pi iz(|x|^2\frac{1}{\nu_1}-1)}e^{-\pi(\epsilon z)^2}dzdx.$$

Then $S(\nu) = \lim_{\epsilon\to 0} I_\epsilon$. Isolating the last part of I_ϵ, we let

$$J_\epsilon = \int_{\mathbb{R}} e^{-2\pi iz(\frac{|x|^2}{\nu_1}-1)}e^{-\pi(\epsilon z)^2}dz.$$

If we let $X = \epsilon z$, $Y = \frac{\frac{|x|^2}{\nu_1} - 1}{\epsilon}$, then

$$J_\epsilon = \frac{1}{\epsilon} \int_{\mathbb{R}} e^{-2\pi i (XY)} e^{-\pi X^2} dX.$$

But this integral is just the Fourier transform of $e^{-\pi X^2}$ evaluated at Y. As noted above, the Fourier transform of $e^{-\pi X^2}$ is itself. So

$$\begin{aligned} J_\epsilon &= \frac{1}{\epsilon} e^{-\pi Y^2} \\ &= \frac{1}{\epsilon} e^{-\pi (\frac{\frac{|x|^2}{\nu_1} - 1}{\epsilon})^2}. \end{aligned}$$

Plugging this back into the definition for I_ϵ gives

$$I_\epsilon = \frac{1}{\nu_1 \epsilon} \int_{\mathbb{R}^n} e^{-\pi |x|^2} \int_{\mathbb{R}} e^{-2\pi i (\sum_{i=1}^n (\lambda_i - \frac{\nu_2}{\nu_1}) x_i^2) \xi_2} d\xi_2 e^{-\pi (\frac{\frac{|x|^2}{\nu_1} - 1}{\epsilon})^2} dx.$$

Next, let us change to hyperspherical coordinates, i.e. let

$$x_1 = t \cos \phi_1,$$

$$x_2 = t \sin \phi_1 \cos \phi_2,$$

$$\dots$$

$$x_{n-1} = t \sin \phi_1 \sin \phi_2 ... \sin \phi_{n-2} \cos \phi_{n-1},$$

$$x_n = t \sin \phi_1 \sin \phi_2 ... \sin \phi_{n-2} \sin \phi_{n-1}.$$

So $|x|^2 = t^2$. For ease of notation, let $\omega_i = \frac{x_i}{t}$. Then

$$\begin{aligned} I_\epsilon =& \frac{1}{\nu_1 \epsilon} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} e^{-\pi t^2} e^{-2\pi i t^2 (\sum_{i=1}^n (\lambda_i - \frac{\nu_2}{\nu_1}) w_i^2) \xi_2} e^{-\pi (\frac{\frac{t^2}{\nu_1} - 1}{\epsilon})^2} t^{n-1} \\ & \cdot \sin^{n-2} \phi_1 \sin^{n-3} \phi_2 ... \sin \phi_{n-2} d\xi_2 dt d\phi. \end{aligned}$$

Let

$$y = \frac{t^2 - \nu_1}{\epsilon \nu_1}.$$

Then

$$y \epsilon \nu_1 + \nu_1 = t^2,$$

$$\epsilon \nu_1 dy = 2t dt.$$

So

$$I_\epsilon = \int_{\mathbb{R}} \int_{S^{n-1}} \int_{-\frac{1}{\epsilon}}^{\infty} e^{-\pi(y\epsilon\nu_1+\nu_1)} e^{-2\pi i(y\epsilon\nu_1+\nu_1)(\sum_{i=1}^{n}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2)\xi_2} e^{-\pi y^2}$$
$$\cdot (y\epsilon\nu_1+\nu_1)^{\frac{n-2}{2}} \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 ... \sin\phi_{n-2} \frac{dy}{2} d\phi d\xi_2.$$

If we take the limit as $\epsilon \to 0$,

$$\begin{aligned}\lim_{\epsilon\to 0} I_\epsilon = \frac{1}{2} \int_{\mathbb{R}} \int_{S^{n-1}} \int_{-\infty}^{\infty} & e^{-\pi(\nu_1)} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2)\xi_2} e^{-\pi y^2} (\nu_1)^{\frac{n-2}{2}} \\ & \cdot \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 ... \sin\phi_{n-2} dy d\phi d\xi_2 \\ = & \frac{1}{2} \int_{\mathbb{R}} \int_{S^{n-1}} e^{-\pi(\nu_1)} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2)\xi_2} (\nu_1)^{\frac{n-2}{2}} \\ & \cdot \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 ... \sin\phi_{n-2} d\phi d\xi_2,\end{aligned}$$

where the last line comes from the well-known fact that

$$\int_{-\infty}^{\infty} e^{-\pi y^2} dy = 1,$$

since this is the integral of the Gaussian distribution over all $\mathbb{R}$. So

$$S(\nu) = \frac{1}{2} e^{-\pi(\nu_1)} (\nu_1)^{\frac{n-2}{2}} \int_{\mathbb{R}} \int_{S^{n-1}} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2)\xi_2} \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 ... \sin\phi_{n-2} d\phi d\xi_2.$$

For future ease, we write out the bounds for integration over S^{n-1} explicitly. If we let

$$K = \frac{1}{2} e^{-\pi(\nu_1)} (\nu_1)^{\frac{n-2}{2}}$$

then

$$\begin{aligned}\frac{S(\nu)}{K} &= \int_{\mathbb{R}} \int_0^{2\pi} \int_0^{\pi} ... \int_0^{\pi} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2)\xi_2} \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 ... \sin\phi_{n-2} d\phi d\xi_2 \\ &= \int_{\mathbb{R}} \int_0^{\pi} \int_0^{\pi} ... \int_0^{\pi} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2)\xi_2} \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 ... \sin\phi_{n-2} d\phi d\xi_2 \\ &\quad + \int_{\mathbb{R}} \int_{\pi}^{2\pi} \int_0^{\pi} ... \int_0^{\pi} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2)\xi_2} \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 ... \sin\phi_{n-2} d\phi d\xi_2.\end{aligned}$$

2.3 Taylor Series

The goal of this section is to rewrite the expression for $S(\nu)$ with the Taylor Series expansion for e; this allows us to take the integral more effectively. First, though, we wish to remove the last variable w_n; to do this, we change from hyperspherical coordinates to rectangular coordinates and then back to hyperspherical.

To this end, let w_i be as above. So

$$w_1 = \cos\phi_1,$$

$$w_2 = \sin\phi_1\cos\phi_2,$$

$$\dots$$

$$w_{n-1} = \sin\phi_1\sin\phi_2 ... \sin\phi_{n-2}\cos\phi_{n-1},$$

$$w_n^2 = 1 - \sum_{i=1}^{n-1} w_i^2.$$

To change variables, we must compute the Jacobian $J|\frac{\partial\phi_i}{\partial w_j}|$. For $1 \leq i \leq n-1$,

$$\frac{\partial\phi_i}{\partial w_i} = -\frac{1}{\sin\phi_1\sin\phi_2 ... \sin\phi_i}.$$

Note that $\frac{\partial\phi_i}{\partial w_j} = 0$ for $i > j$, so the Jacobian matrix is upper triangular. Thus, if we change variables from ϕ to w, we see that the change of measure is

$$\begin{aligned} J|\frac{\partial\phi_i}{\partial w_j}| &= |\prod_{i=1}^{n-1}\frac{\partial\phi_i}{\partial w_i}| \\ &= \frac{1}{\sin^{n-1}\phi_1\sin^{n-2}\phi_2 ... \sin\phi_{n-1}} \\ &= \frac{1}{|w_n|}\frac{1}{\sin^{n-2}\phi_1\sin^{n-3}\phi_2 ... \sin\phi_{n-2}}. \end{aligned}$$

Then

$$\frac{S(\nu)}{K} = \int_{\mathbb{R}} \int_{-1}^{1} \int_{-1+w_1^2}^{1-w_1^2} ....$$

$$... \int_{-1+\sum_{i=1}^{n-1} w_i^2}^{1-\sum_{i=1}^{n-1} w_i^2} \frac{1}{|w_n|} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2 + (\lambda_n - \frac{\nu_2}{\nu_1})(1-\sum_{i=1}^{n-1} w_i^2))\xi_2} dw_{n-1}...dw_1 d\xi_2$$

$$+ \int_{\mathbb{R}} \int_{-1}^{1} \int_{-1+w_1^2}^{1-w_1^2} ...$$

$$... \int_{-1+\sum_{i=1}^{n-1} w_i^2}^{1-\sum_{i=1}^{n-1} w_i^2} \frac{1}{|w_n|} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2 + (\lambda_n - \frac{\nu_2}{\nu_1})(1-\sum_{i=1}^{n-1} w_i^2))\xi_2} dw_{n-1}...dw_1 d\xi_2,$$

where we must split the integrals separately into $0 \leq \phi_{n-1} < \pi$ and $\pi \leq \phi_{n-1} < 2\pi$ so that the change of variables is an injection and hence is well-defined. So the above is

$$2 \int_{\mathbb{R}} \int_{-1}^{1} \int_{-1+w_1^2}^{1-w_1^2} ...$$

$$... \int_{-1+\sum_{i=1}^{n-1} w_i^2}^{1-\sum_{i=1}^{n-1} w_i^2} \frac{1}{|w_n|} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2 + (\lambda_n - \frac{\nu_2}{\nu_1})(1-\sum_{i=1}^{n-1} w_i^2))\xi_2} dw_{n-1}...dw_1 d\xi_2$$

$$= 2 \int_{\mathbb{R}} \int_{-1}^{1} \int_{-1+w_1^2}^{1-w_1^2} ...$$

$$... \int_{-1+\sum_{i=1}^{n-1} w_i^2}^{1-\sum_{i=1}^{n-1} w_i^2} \frac{1}{\sqrt{1-\sum_{i=1}^{n-1} w_i^2}} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \frac{\nu_2}{\nu_1})w_i^2 + (\lambda_n - \frac{\nu_2}{\nu_1})(1-\sum_{i=1}^{n-1} w_i^2))\xi_2} dw_{n-1}...dw_1 d\xi_2...$$

$$= 2 \int_{\mathbb{R}} \int_{-1}^{1} \int_{-1+w_1^2}^{1-w_1^2}$$

$$... \int_{-1}^{1} \frac{1}{\sqrt{1-\sum_{i=1}^{n-1} w_i^2}} e^{-2\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)w_i^2 + (\lambda_n - \frac{\nu_2}{\nu_1}))\xi_2} dw_{n-1}...dw_1 d\xi_2.$$

Now, we will put this back into hyperspherical coordinates, except in one fewer variable. So

$$w_1 = t \cos \phi_1,$$

$$w_2 = t \sin \phi_1 \cos \phi_2,$$

$$\cdots$$

$$w_{n-2} = t \sin \phi_1 \sin \phi_2 ... \sin \phi_{n-3} \cos \phi_{n-2},$$

$$w_{n-1} = t \sin \phi_1 \sin \phi_2 ... \sin \phi_{n-3} \sin \phi_{n-2}.$$

For notational convenience, we write

$$t = \sqrt{\sum_{i=1}^{n-1} w_i^2},$$

$$u_i = \frac{w_i}{t},$$

$$dV = \sin^{n-3}\phi_1 \sin^{n-4}\phi_2 ... \sin\phi_{n-3} du.$$

Then we have

$$2\int_{\mathbb{R}}\int_{S^{n-2}}\int_0^1 \frac{1}{\sqrt{1-t^2}} e^{-2\pi i(\nu_1)(t^2\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2+(\lambda_n-\frac{\nu_2}{\nu_1}))\xi_2} dt dV d\xi_2$$
$$= 2\int_{S^{n-2}}\int_{\mathbb{R}} e^{-2\pi i\nu_1(\lambda_n-\frac{\nu_2}{\nu_1})\xi_2} \int_0^1 \frac{1}{\sqrt{1-t^2}} e^{-2\pi i(\nu_1)(t^2\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2)\xi_2} dt d\xi_2 dV.$$

Examining only the integral over dt, we write the exponential in terms of its Taylor series expansion:

$$\int_0^1 \frac{1}{\sqrt{1-t^2}} \sum_{k=0}^{\infty} \frac{(-2\pi i(\nu_1)(t^2\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2\xi_2))^k}{k!} dt,$$

where we have used here the fact that

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

Now, the sum is absolutely convergent, and the integral is over a finite interval. Thus, we can rearrange summation and integration and find the integral of each summand. So the above is equal to

$$\sum_{k=0}^{\infty}\int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{(-2\pi i(\nu_1)(t^2\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2\xi_2))^k}{k!} dt$$
$$= \sum_{k=0}^{\infty} \frac{(-2\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2\xi_2))^k}{k!} \int_0^1 \frac{t^{2k}}{\sqrt{1-t^2}} dt.$$

For this integral, we will integrate by parts repeatedly and then use a trigonometric substitution. Performing the integration by parts first, we let

$$u = t^{2k-1},$$

$$du = (2k-1)t^{2k-2}dt,$$

$$v = \sqrt{1-t^2},$$

$$dv = \frac{t}{\sqrt{1-t^2}}dt.$$

So

$$\begin{aligned}
&\int_0^1 \frac{t^{2k}}{\sqrt{1-t^2}}dt \\
&\quad = (\sqrt{1-t^2})t^{2k-1}]_0^1 - \int_0^1 (2k-1)t^{2k-2}\sqrt{1-t^2}dt \\
&\quad = -\int_0^1 (2k-1)t^{2k-2}\sqrt{1-t^2}dt.
\end{aligned}$$

Again using integration by parts, this time with

$$v = \frac{1}{3}(\sqrt{1-t^2})^3$$

$$dv = t\sqrt{1-t^2}dt$$

$$u = -(2k-1)t^{2k-3}$$

$$du = -(2k-1)(2k-3)t^{2k-4}.$$

we see that the above integral is

$$\begin{aligned}
&-\frac{(2k-1)}{3}(\sqrt{1-t^2})^3 t^{2k-3}]_0^1 + \frac{1}{3}\int_0^1 (2k-1)(2k-3)t^{2k-4}(\sqrt{1-t^2})^3 dt \\
&\quad = \frac{(2k-1)(2k-3)}{3}\int_0^1 t^{2k-4}(\sqrt{1-t^2})^3 dt.
\end{aligned}$$

Continuing this process iteratively, we finally find that

$$\begin{aligned}
&\int_0^1 \frac{t^{2k}}{\sqrt{1-t^2}}dt \\
&\quad = \frac{(2k-1)(2k-3)...(1)}{(1)(3)(5)...(2k-1)}(-1)^k \int_0^1 (\sqrt{1-t^2})^{2k-1}dt \\
&\quad = (-1)^k \int_0^1 (\sqrt{1-t^2})^{2k-1}dt.
\end{aligned}$$

Now, to evaluate this, we require a trigonometric substitution:

$$t = \sin u,$$

$$dt = \cos u du.$$

Then

$$\begin{aligned}
&\int_0^1 (\sqrt{1-t^2})^{2k-1} dt \\
&= \int_0^{\frac{\pi}{2}} (\cos u)^{2k-1} \cos u du \\
&= \int_0^{\frac{\pi}{2}} (\cos u)^{2k} du \\
&= \frac{1}{2k}(\cos u)^{2k-1} \sin u]_0^{\frac{\pi}{2}} + \frac{2k-1}{2k} \int_0^{\frac{\pi}{2}} (\cos u)^{2k-2} du \\
&= \frac{2k-1}{2k} \int_0^{\frac{\pi}{2}} (\cos u)^{2k-2} du.
\end{aligned}$$

where the integration here is again by parts. Following this process iteratively, we get

$$\begin{aligned}
&\int_0^1 (\sqrt{1-t^2})^{2k-1} dt \\
&= \frac{(2k-1)(2k-3)...(1)}{2k(2k-2)...(2)} \int_0^{\frac{\pi}{2}} du \\
&= \frac{(2k)!}{(2^k(k!))^2}(\frac{\pi}{2}).
\end{aligned}$$

So our sum is

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(-2\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)u_i^2 \xi_2))^k}{k!} \frac{(2k)!}{(2^k(k!))^2}(\frac{\pi}{2}) \\
&= (\frac{\pi}{2}) \sum_{k=0}^{\infty} (\frac{-2\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)u_i^2 \xi_2)}{4})^k \frac{(2k)!}{(k!)^3} \\
&= (\frac{\pi}{2}) \sum_{k=0}^{\infty} (\frac{-\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)u_i^2 \xi_2)}{2})^k \frac{(2k)!}{(k!)^3}.
\end{aligned}$$

2.4 Bessel Functions

In this chapter, we review the applicable basic properties of Bessel functions.* Consider the differential equation

$$x^2y'' + xy' + x^2y = 0.$$

This is known as the 0-th Bessel differential equation. It is an easy exercise to show that one solution to this differential equation, which we will define to be $J_0(x)$, is

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!^2}.$$

Another solution, denoted $Y_0(x)$, is

$$Y_0(x) = J_0(x)\log x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^2k}{(2k)!^2}(\sum_{j=1}^{k}\frac{1}{j}).$$

$J_0(x)$ is known as the 0-th Bessel function, and Y_0 is known as the 0-th modified Bessel-Neumann function. Importantly, any other solution to this differential equation is a linear combination of J_0 and Y_0, so if $y(x)$ is a solution then

$$y(x) = a_1J_0(x) + a_2Y_0(x)$$

for constants a_1, a_2. Note that

$$\lim_{x\to 0} Y_0(x) = -\infty,$$

$$\lim_{x\to 0} J_0(x) = 1.$$

So if $\lim_{x\to 0} y(x) = 1$ then a_2 must be zero and a_1 must be 1. Hence if $\lim_{x\to 0} y(x) = 1$ then $y(x) = J_0(x)$.

Recall from last section that we were left with the sum

$$(\frac{\pi}{2})\sum_{k=0}^{\infty}(\frac{-\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)u_i^2\xi_2)}{2})^k\frac{(2k)!}{(k!)^3}.$$

*For a more expansive overview, one can examine [1]

We show that this is related to the J_0 described above; the following lemma will help illustrate the relationship:

Lemma 2.4.1. *Let $J_0(x)$ denote the 0-th Bessel function. Then*

$$\sum_{k=0}^{\infty}(ix)^k\frac{(2k)!}{k!^3} = J_0(2x)e^{2ix}.$$

Proof. First, note the equivalence between this statement and

$$e^{-ix}\sum_{k=0}^{\infty}(\frac{ix}{2})^k\frac{(2k)!}{k!^3} = J_0(x).$$

Let

$$y = e^{-ix}\sum_{k=0}^{\infty}(\frac{ix}{2})^k\frac{(2k)!}{k!^3}.$$

The lemma will be proven in two steps; we show that y solves the Bessel Differential Equation above, and then we prove that $\lim_{x\to 0} y = 1$. From above, this implies the lemma.

In order to show that y solves the necessary differential equation, we must first compute the various derivatives for y:

$$y' = -ie^{-ix}\sum_{k=0}^{\infty}(\frac{ix}{2})^k\frac{(2k)!}{k!^3} + \frac{1}{2}ie^{-ix}\sum_{k=1}^{\infty}k(\frac{ix}{2})^{k-1}\frac{(2k)!}{k!^3}$$

$$y'' = -e^{-ix}\sum_{k=0}^{\infty}(\frac{ix}{2})^k\frac{(2k)!}{k!^3} + e^{-ix}\sum_{k=1}^{\infty}k(\frac{ix}{2})^{k-1}\frac{(2k)!}{k!^3} - \frac{1}{4}e^{-ix}\sum_{k=2}^{\infty}k(k-1)(\frac{ix}{2})^{k-2}\frac{(2k)!}{k!^3}.$$

From here, we must plug y and its derivatives into the Bessel Differential equation and show that it is satisfied; this is merely an exercise in basic algebra:

$$
\begin{aligned}
&x^2y'' + xy' + x^2y \\
&\quad = -4(\frac{ix}{2})^2y'' + \frac{2}{i}(\frac{ix}{2})y' - 4(\frac{ix}{2})^2y \\
&\quad = -4(\frac{ix}{2})^2(-e^{-ix}\sum_{k=0}^{\infty}(\frac{ix}{2})^k\frac{(2k)!}{k!^3} + e^{-ix}\sum_{k=1}^{\infty}k(\frac{ix}{2})^{k-1}\frac{(2k)!}{k!^3} \\
&\qquad - \frac{1}{4}e^{-ix}\sum_{k=2}^{\infty}k(k-1)(\frac{ix}{2})^{k-2}\frac{(2k)!}{k!^3}) \\
&\qquad + \frac{2}{i}(\frac{ix}{2})(-ie^{-ix}\sum_{k=0}^{\infty}(\frac{ix}{2})^k\frac{(2k)!}{k!^3} + \frac{1}{2}ie^{-ix}\sum_{k=1}^{\infty}k(\frac{ix}{2})^{k-1}\frac{(2k)!}{k!^3}) \\
&\qquad - 4(\frac{ix}{2})^2e^{-ix}\sum_{k=0}^{\infty}(\frac{ix}{2})^k\frac{(2k)!}{k!^3} \\
&\quad = -4e^{-ix}\sum_{k=1}^{\infty}k(\frac{ix}{2})^{k+1}\frac{(2k)!}{k!^3} + e^{-ix}\sum_{k=2}^{\infty}k(k-1)(\frac{ix}{2})^k\frac{(2k)!}{k!^3} \\
&\qquad - 2e^{-ix}\sum_{k=0}^{\infty}(\frac{ix}{2})^{k+1}\frac{(2k)!}{k!^3} + e^{-ix}\sum_{k=1}^{\infty}k(\frac{ix}{2})^k\frac{(2k)!}{k!^3} \\
&\qquad - 4e^{-ix}\sum_{k=0}^{\infty}k(\frac{ix}{2})^{k+1}\frac{(2k)!}{k!^3} + e^{-ix}\sum_{k=1}^{\infty}k(k-1)(\frac{ix}{2})^k\frac{(2k)!}{k!^3} \\
&\qquad - 2e^{-ix}\sum_{k=0}^{\infty}(\frac{ix}{2})^{k+1}\frac{(2k)!}{k!^3} + e^{-ix}\sum_{k=1}^{\infty}k(\frac{ix}{2})^k\frac{(2k)!}{k!^3} \\
&\quad = -2e^{-ix}\sum_{k=0}^{\infty}(2k+1)(\frac{ix}{2})^{k+1}\frac{(2k)!}{k!^3} + e^{-ix}\sum_{k=1}^{\infty}k(k-1+1)(\frac{ix}{2})^k\frac{(2k)!}{k!^3} \\
&\quad = -2e^{-ix}\sum_{k=0}^{\infty}(2k+1)(\frac{ix}{2})^{k+1}\frac{(2k)!}{k!^3} + e^{-ix}\sum_{k=0}^{\infty}(k+1)^2(\frac{ix}{2})^{k+1}\frac{(2(k+1))!}{(k+1)!^3} \\
&\quad = -2e^{-ix}\sum_{k=0}^{\infty}(2k+1)(\frac{ix}{2})^{k+1}\frac{(2k)!}{k!^3} \\
&\qquad + e^{-ix}\sum_{k=0}^{\infty}(k+1)^2(2k+2)(2k+1)(\frac{ix}{2})^{k+1}\frac{(2k)!}{(k+1)^3(k!)^3} \\
&\quad = -2e^{-ix}\sum_{k=0}^{\infty}(2k+1)(\frac{ix}{2})^{k+1}\frac{(2k)!}{k!^3} \\
&\qquad + e^{-ix}\sum_{k=0}^{\infty}2(2k+1)(\frac{ix}{2})^{k+1}\frac{(2k)!}{(k!)^3} \\
&\quad = 0.
\end{aligned}
$$

So y satisfies the differential equation. Additionally, it is easy to see that

$$\lim_{x\to 0} e^{-ix}\sum_{k=0}^{\infty}(\frac{ix}{2})^k\frac{(2k)!}{k!^3} = 1.$$

Thus, $y = J_0(x)$. □

As a result of this lemma, we have

$$\begin{aligned}(\frac{\pi}{2})\sum_{k=0}^{\infty}(&\frac{-\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2\xi_2)}{2})^k\frac{(2k)!}{(k!)^3}\\ &= (\frac{\pi}{2})J_0(-\pi(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2\xi_2))e^{(-\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2\xi_2))}.\end{aligned}$$

2.5 Fourier Transforms

Using the findings from the previous chapter, we first rewrite $S(\nu)$:

$$\begin{aligned}\frac{S(\nu)}{K} =& 2\frac{\pi}{2}\int_{S^{n-2}}\int_{\mathbb{R}} J_0(-\pi(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2\xi_2))\\ &\cdot e^{(-\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2\xi_2))}e^{-2\pi i\nu_1(\lambda_n-\frac{\nu_2}{\nu_1})\xi_2}d\xi_2 dV\\ =& \pi\int_{S^{n-2}}\int_{\mathbb{R}} J_0(-\pi(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2\xi_2))e^{(-\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2)+2(\lambda_n-\frac{\nu_2}{\nu_1}))\xi_2}d\xi_2 dV.\end{aligned}$$

Note that the inner integral is the Fourier transform of

$$J_0(-\pi(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2))$$

evaluated at

$$\frac{\nu_1((\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2)+2(\lambda_n-\frac{\nu_2}{\nu_1}))}{2}.$$

This type of quantity has been computed classically; here, we state the result of this computation without proof†:

$$\int_{\mathbb{R}} J_0(t)e^{-2\pi ixt}dt = \frac{2\cdot\mathrm{rect}(\pi x)}{\sqrt{1-4\pi^2x^2}}$$

†For a reference about this result, see [2]

where rect is the rectangular function given by

$$\text{rect}(t) = \begin{cases} 0 & \text{if } |t| > \frac{1}{2}, \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2}, \\ 1 & \text{if } |t| < \frac{1}{2}. \end{cases}$$

So if we let

$$x = \frac{(\nu_1)((\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)u_i^2) + (\lambda_n - \frac{\nu_2}{\nu_1}))}{2},$$

$$k = -\pi(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)u_i^2),$$

$$w = k\xi_2,$$

$$dw = kd\xi_2,$$

then

$$\begin{aligned}
&\int_{\mathbb{R}} J_0(-\pi(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i - \lambda_n)u_i^2\xi_2))e^{(-\pi i(\nu_1)(\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2)+2(\lambda_n-\frac{\nu_2}{\nu_1}))\xi_2}d\xi_2 \\
&= \int_{\mathbb{R}} J_0(k\xi_2)e^{(-2\pi ix)\xi_2}d\xi_2 \\
&= \frac{1}{|k|}\int_{\mathbb{R}} J_0(w)e^{(-2\pi ix)\frac{w}{k}}dw \\
&= \frac{2\cdot\text{rect}(\pi\frac{x}{k})}{|k|\sqrt{1-4\pi^2(\frac{x}{k})^2}}.
\end{aligned}$$

where we have $|k|$ because if $k < 0$ then the change of variables causes the bounds to go from ∞ to $-\infty$, and reorienting the bounds cancels the negative in front of k.

Thus,

$$\frac{S(\nu)}{K} = \pi\int_{S^{n-2}} \frac{2\cdot\text{rect}(\pi\frac{x}{k})}{|k|\sqrt{1-4\pi^2(\frac{x}{k})^2}}dV.$$

Plugging in x and k, we get

$$\frac{S(\nu)}{K} = \pi\int_{S^{n-2}} \frac{2\cdot\text{rect}(\frac{1}{2}+\frac{(\lambda_n-\frac{\nu_2}{\nu_1})}{\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2})}{|\pi(\nu_1)\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2|\sqrt{1-(1+\frac{2(\lambda_n-\frac{\nu_2}{\nu_1})}{\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)u_i^2})^2}}dV.$$

which is as stated in Theorem 2.

Chapter 3

$v \neq \infty$: $f = (f_1, f_2)$ over $\mathbb{Q}_p$

3.1 Gauss Sums

Throughout the upcoming sections, we will use extensively the classical calculation of Gauss sums. As such, we first show how such a calculation is performed. Here, the actual method given is that of Siegel, and the result is a generalization of that of Gauss.

Let $a, b \in \mathbb{N}$, $\lambda \in \mathbb{Q}$ be such that

$$ab + 2a\lambda \equiv 0 \pmod 2.$$

We will evaluate the sum

$$\frac{1}{\sqrt{b}} \sum_{h=1}^{b} e^{\pi i \frac{a}{b} h^2 + \pi i a h}$$

(cf. [7], p. 50). This is actually more general than is necessary; for our purposes, it will be sufficient to let $a = 2r$, $b = p^l$, which gives the sum

$$p^{-\frac{l}{2}} \sum_{x=1}^{p^l} e^{\frac{2\pi i}{p^l} x^2 r}.$$

Next, we let $z \in \mathbb{C}$ and define

$$f(z) = \frac{e^{\pi i \frac{a}{b}(z+\lambda)^2}}{e^{2\pi i z} - 1}.$$

Note that the numerator is never zero, and the denominator has a pole at every integer h (and at no other points). We show that the poles are all of order 1:

$$\begin{aligned} Res_{z=h} f(z) &= \lim_{z \to h} (z-h) f(z) \\ &= e^{\pi i \frac{a}{b}(h+\lambda)^2} \lim_{z \to h} \frac{(z-h)}{e^{2\pi i z} - 1} \\ &= e^{\pi i \frac{a}{b}(h+\lambda)^2} \lim_{z \to h} \frac{1}{2\pi i e^{2\pi i z}} \\ &= e^{\pi i \frac{a}{b}(h+\lambda)^2} \frac{1}{2\pi i} \end{aligned}$$

where the penultimate step is by L'Hospital's rule. Thus, if we have a contour C which contains the integers $1, 2, ..., b$ (and no others) then

$$\int_C f(z) dz = 2\pi i \sum_{h=1}^{b} Res_{z=h} f(z) = \sum_{h=1}^{b} e^{\pi i \frac{a}{b}(h+\lambda)^2}.$$

In particular, we will take C to be the contour in Figure 2, where L is the line which goes through the real axis at a 45° angle and L_0 is the line parallel to L which goes through the real axis at $b + \frac{1}{2}$. We claim that as the horizontal lines on the top and bottom of the contour go toward infinity, the integrals over these lines go to zero. This is because

$$f(x+iy) = \frac{e^{\pi i \frac{a}{b}((x+\lambda)^2 + 2iy(x+\lambda) - y^2)}}{e^{2\pi i x - 2\pi y} - 1}.$$

So $f(z) \to 0$ if $e^{-\pi \frac{a}{b}(2y(x+\lambda))} \to 0$ and $e^{2\pi i x - 2\pi y} \not\to 1$. For the upper horizontal line, $y \to \infty$ and $x + \lambda$ is positive; for the lower horizontal line, $y \to -\infty$ and $x + \lambda$ is negative. Thus, in both cases, $e^{-\pi \frac{a}{b}(2y(x+\lambda))} \to 0$ and $e^{2\pi i x - 2\pi y} \not\to 1$, and hence $f(z) \to 0$.

Now, note that if z is on the line L then $z + b$ is on the line L_0. So

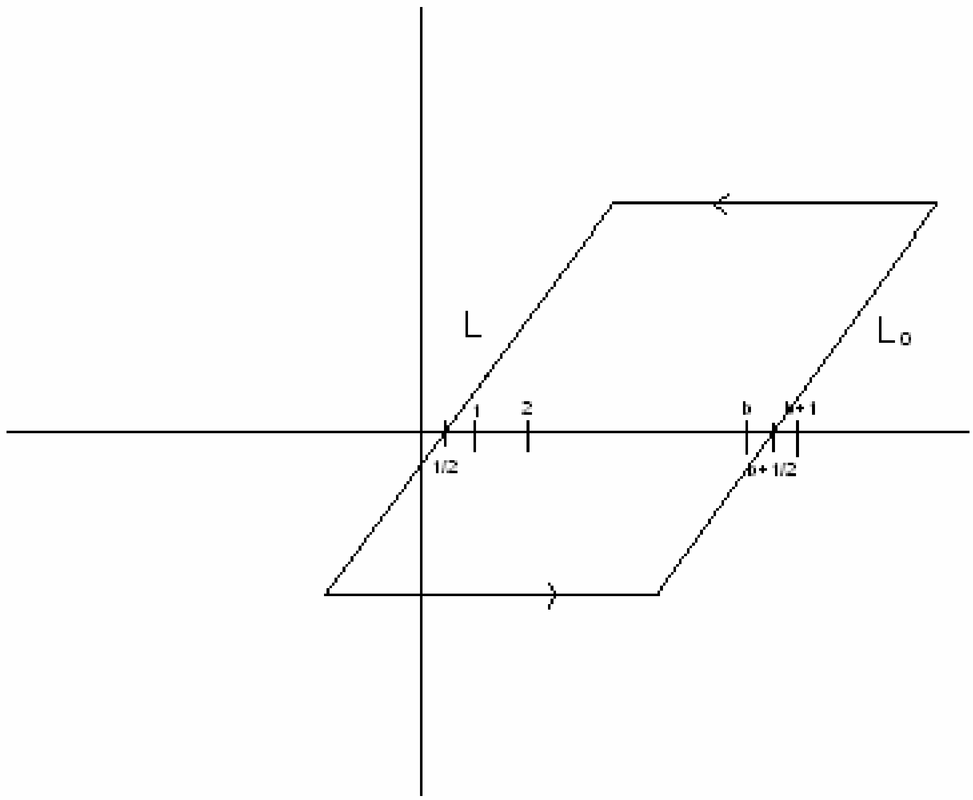

Figure 3.1: Contour of integration

$$\begin{aligned}\int_C f(z)dz &= \int_{L_0} f(z)dz - \int_L f(z)dz \\ &= \int_L (f(z+b) - f(z))dz \\ &= \int_L \frac{e^{\pi i \frac{a}{b}(z+\lambda)^2}}{e^{2\pi i z} - 1}[e^{\pi i \frac{a}{b}(2b(z+\lambda)+b^2)} - 1]dz.\end{aligned}$$

Now,

$$\frac{a}{b}(2b(z+\lambda) + b^2) = 2a\lambda + ab + 2az \equiv 2az \pmod 2$$

by our assumption earlier, which means that

$$e^{\pi i \frac{a}{b}(2b(z+\lambda)+b^2)} = e^{2\pi i a z}.$$

Then

$$\begin{aligned}\int_C f(z)dz &= \int_L e^{\pi i \frac{a}{b}(z+\lambda)^2}[\frac{e^{\pi i \frac{a}{b}(2b(z+\lambda)+b^2)} - 1}{e^{2\pi i z} - 1}]dz \\ &= \int_L e^{\pi i \frac{a}{b}(z+\lambda)^2}[\sum_{h=0}^{a-1} e^{2\pi i h z}]dz \\ &= \sum_{h=0}^{a-1} \int_L e^{\pi i \frac{a}{b}(z+\lambda)^2 + 2\pi i h z} dz \\ &= \sum_{h=0}^{a-1} e^{\pi i \frac{b}{a} h^2 - 2\pi i h \lambda} \int_L e^{\pi i \frac{a}{b}(z+\lambda+\frac{b}{a}h)^2} dz.\end{aligned}$$

Here, the first step comes from the expansion

$$\frac{x^a - 1}{x - 1} = \sum_{h=0}^{a-1} x^h.$$

Let us define

$$Z = z + \lambda + \frac{b}{a}h.$$

Then the line L instead shifts to another parallel line which we can denote L_1, i.e.

$$\sum_{h=0}^{a-1} e^{\pi i \frac{b}{a} h^2 - 2\pi i h \lambda} \int_{L_1} e^{\pi i \frac{a}{b} Z^2} dZ.$$

Let L' be the line parallel to L_1 which goes through the origin. Since the integrand above has no poles, it follows from Cauchy's Theorem that we can shift the contour of integration from L_1 to L'. The integral in the above expression is then

$$\int_{L'} e^{\pi i \frac{a}{b} Z^2} dZ.$$

We make the change of variables

$$Z = -\sqrt{\frac{ib}{a}} t$$
$$dZ = -\sqrt{\frac{ib}{a}} dt.$$

Note that the integral over L', like the integrals over L and L_1, goes downward along the line. So the change of measure allows us to change the orientation of the integral, i.e.

$$\int_{L'} e^{\pi i \frac{a}{b} Z^2} dZ = \sqrt{\frac{ib}{a}} \int_{-\infty}^{\infty} e^{-\pi t^2} dZ = \sqrt{\frac{ib}{a}}.$$

Thus, in summary,

$$\begin{aligned}\int_C f(z)dz &= \sum_{h=1}^{b} e^{\pi i \frac{a}{b}(h+\lambda)^2} \\ &= \sqrt{\frac{ib}{a}} \sum_{h=0}^{a-1} e^{-\pi i \frac{b}{a} h^2 - 2\pi i h \lambda} \\ &= \sqrt{\frac{ib}{a}} \sum_{h=1}^{a} e^{-\pi i \frac{b}{a} h^2 - 2\pi i h \lambda}\end{aligned}$$

where the shift in parameter in the last step is allowed because

$$ab + 2a\lambda \equiv 0 \pmod{2}.$$

We arrive, then, at the theorem

Theorem 3.1.1. *Let a, b, and λ be as above. Then*

$$\sqrt{\frac{1}{b}} \sum_{x=1}^{b} e^{\pi i \frac{a}{b}(x+\lambda)^2} = \sqrt{\frac{i}{a}} \sum_{x=1}^{a} e^{\pi i \frac{b}{a} x^2 - 2\pi i x \lambda}.$$

As a corollary, we can prove the special case which will be of use to us throughout the remainder of the chapter:

Corollary 3.1.2. *Let $a = 2$, $\lambda = 0$, $b = p$ be a prime number. Then*

$$\frac{1}{p}\sum_{x=1}^{p} e^{2\pi i x^2 \frac{1}{p}} = \begin{cases} \frac{1}{\sqrt{p}} & \textit{if } p \equiv 1 \ (\textit{mod } 4), \\ \frac{i}{\sqrt{p}} & \textit{if } p \equiv 3 \ (\textit{mod } 4), \\ 0 & \textit{if } p = 2. \end{cases}$$

Proof. We know that

$$\sqrt{i} = e^{\frac{i\pi}{4}} = \frac{1+i}{\sqrt{2}}.$$

Then

$$\frac{1}{\sqrt{p}}\sum_{x=1}^{p} e^{2\pi i x^2 \frac{1}{p}} = \frac{1}{\sqrt{2}}\frac{(1+i)}{\sqrt{2}}(1 + e^{-\pi i \frac{p}{2}}).$$

Naturally, $e^{-\pi i \frac{p}{2}}$ depends on p. In particular,

$$1 + e^{-\pi i \frac{p}{2}} = \begin{cases} 1 - i & \text{if } p \equiv 1 \pmod 4, \\ 1 + i & \text{if } p \equiv 3 \pmod 4, \\ 0 & \text{if } p = 2. \end{cases}$$

from which the corollary follows. □

Now, we can derive a useful theorem about Legendre characters. For an odd prime p, let us choose a $\zeta \in \widehat{\mathbb{F}_p^+} = Hom(\mathbb{F}_p, S^1)$ such that

$$\widehat{\mathbb{F}_p^+} = <\zeta> .$$

We define

$$\begin{aligned} R(\zeta) &= \sum_{x \in \mathbb{F}_p} \zeta^{x^2} \\ &= 1 + \sum_{x \in \mathbb{F}_p^\times} \zeta^{x^2} \\ &= 1 + \sum_{x<0} \zeta^{x^2} + \sum_{x>0} \zeta^{x^2}, \end{aligned}$$

where an element in $\mathbb{F}_p^\times$ can be described as being > 0 if it is between 0 and $\frac{p-1}{2}$ or < 0 otherwise. Then

$$\begin{aligned} R(\nu) &= 1 + 2\sum_{x>0} \zeta^{x^2} \\ &= 1 + 2\sum_{y\in(\mathbb{F}_p^\times)^2} \zeta^y. \end{aligned}$$

If we define

$$\begin{aligned} 0 = &T(\zeta) = \sum_{y\in\mathbb{F}_p} \zeta^y \\ &= 1 + \sum_{y\in\mathbb{F}_p^\times} \zeta^y \\ &= 1 + \sum_{y\in(\mathbb{F}_p^\times)^2} \zeta^y + \sum_{y\in\mathbb{F}_p^\times, y\notin(\mathbb{F}_p^\times)^2} \zeta^y \end{aligned}$$

then we can express $R(\zeta)$ as

$$\begin{aligned} R(\zeta) &= R(\zeta) - T(\zeta) \\ &= \sum_{y\in(\mathbb{F}_p^\times)^2} \zeta^y - \sum_{y\in\mathbb{F}_p^\times, y\notin(\mathbb{F}_p^\times)^2} \zeta^y \\ &= \sum_{y\in\mathbb{F}_p} (\frac{y}{p})\zeta^y, \end{aligned}$$

where $(\frac{y}{p})$ is the Legendre symbol with $(\frac{y}{p}) = 0$ if $p|y$. Thus, we arrive at the following:

Theorem 3.1.3. $\displaystyle\sum_{y\in\mathbb{F}_p} (\frac{y}{p})\zeta^y = \sum_{x=0}^{p-1} \zeta^{x^2}.$

Now, if $p \nmid r$ then we can replace ζ by ζ^r. Then, by the theorem,

$$\begin{aligned} \sum_{x=0}^{p-1} \zeta^{rx^2} &= \sum_{x\in\mathbb{F}_p} (\frac{x}{p})\zeta^{rx} \\ &= (\frac{r}{p}) \sum_{x\in\mathbb{F}_p} (\frac{rx}{p})\zeta^{rx} \\ &= (\frac{r}{p}) \sum_{x=0}^{p-1} \zeta^{x^2}. \end{aligned}$$

Throughout the chapter, this fact will be applied in the following form. Let

$$\zeta^x = \chi(x).$$

Then

$$\sum_{x=0}^{p-1} \chi(rx^2) = (\frac{r}{p}) \sum_{x=0}^{p-1} \chi(x^2).$$

3.2 A p-adic Application of the Gauss Sum

Now that we have our Gauss sum computation, we can relate it to the singular series. Throughout this section, for $a \in \mathbb{Z}_p - p\mathbb{Z}_p$, $(\frac{a}{p})$ will be used to denote the Legendre symbol (similarly, $(\frac{a}{p^2})$ will denote the Jacobi symbol). Of course, we know that if we define $\tilde{a}$ to be $a \pmod p$ then

$$(\frac{a}{p}) = (\frac{\tilde{a}}{p}).$$

This remark augurs an important motif of this section: in many instances, the only term in the p-adic expansion of a which matters is the first one. One such example of this is given in the following lemma, which will be used repeatedly to relate classical Gauss sums to the singular series computations:

Lemma 3.2.1. *Let $p \neq 2$, and assume $a \in \mathbb{Z}_p - p\mathbb{Z}_p$. Moreover, let*

$$G = \begin{cases} \frac{1}{\sqrt{p}} & \text{if } p \equiv 1 \ (mod\ 4), \\ \frac{i}{\sqrt{p}} & \text{if } p \equiv 3 \ (mod\ 4). \end{cases}$$

be the Gauss sum from Corollary 3.1.1. Then

$$\int_{\mathbb{Z}_p} \chi(\frac{a}{p^e} x^2) dx = \begin{cases} p^{-\frac{e}{2}} & \text{if } e \text{ is even}, \\ p^{-\frac{e-1}{2}} (\frac{a}{p}) G & \text{if } e \text{ is odd}. \end{cases}$$

Proof. For ease of notation, we will define

$$A(e,a) = \int_{\mathbb{Z}_p} \chi(\frac{a}{p^e}x^2)dx.$$

We proceed by induction. If $e = 1$ then we have

$$\int_{\mathbb{Z}_p} \chi(\frac{a}{p}x^2)dx = \sum_{b\in\mathbb{F}_p} \int_{b+p\mathbb{Z}_p} \chi(\frac{a}{p}x^2)dx.$$

Now, we change variables, letting $x = b+py$, where y ranges over $\mathbb{Z}_p$. Then $dx = \frac{1}{p}dy$. So the above is

$$\begin{aligned}
&\frac{1}{p}\sum_{b\in\mathbb{F}_p} \int_{\mathbb{Z}_p} \chi(\frac{a}{p}b^2 + \frac{a}{p}(2bpy + p^2y^2))dy \\
&\quad =\frac{1}{p}\sum_{b\in\mathbb{F}_p} \int_{\mathbb{Z}_p} \chi(\frac{a}{p}b^2)\chi(a(2by + py^2))dy \\
&\quad =\frac{1}{p}\sum_{b\in\mathbb{F}_p} \chi(\frac{a}{p}b^2) \int_{\mathbb{Z}_p} dy \\
&\quad =(\frac{a}{p})G,
\end{aligned}$$

where the penultimate step comes from the fact that χ is trivial on $\mathbb{Z}_p$, and the last step is merely the evaluation of a Gauss sum.

For $e = 2$, we will change variables, letting $x = b + py$, where y ranges over $\mathbb{Z}_p$ and b ranges over $\mathbb{F}_p$. So

$$\begin{aligned}
\int_{\mathbb{Z}_p} \chi(\frac{a}{p^2}x^2)dx &= \sum_{b\in\mathbb{F}_p} \int_{b+p\mathbb{Z}_p} \chi(\frac{a}{p^2}x^2)dx \\
&= \sum_{b\in\mathbb{F}_p} \frac{1}{p} \int_{\mathbb{Z}_p} \chi(\frac{a}{p^2}(b+py)^2)dy \\
&= \sum_{b\in\mathbb{F}_p} \frac{1}{p} \int_{\mathbb{Z}_p} \chi(\frac{a}{p^2}(b^2 + 2bpy + p^2y^2))dy \\
&= \sum_{b\in\mathbb{F}_p} \frac{1}{p}\chi(\frac{a}{p^2}(b^2)) \int_{\mathbb{Z}_p} \chi(\frac{a}{p^2}(2bpy + p^2y^2))dy \\
&= \sum_{b\in\mathbb{F}_p} \frac{1}{p}\chi(\frac{a}{p^2}(b^2)) \int_{\mathbb{Z}_p} \chi(\frac{a}{p}2by)dy.
\end{aligned}$$

If $b \neq 0$ then the integral is zero (since the integral of a non-trivial character over $\mathbb{Z}_p$ is zero); otherwise, it's 1. So the above is $\frac{1}{p}$.

Now, if $e > 2$, we can again let $x = b + py$. So

$$\begin{aligned}\int_{\mathbb{Z}_p} \chi(\frac{a}{p^e}x^2)dx &= \sum_{b\in\mathbb{F}_p} \int_{b+p\mathbb{Z}_p} \chi(\frac{a}{p^e}x^2)dx \\ &= \sum_{b\in\mathbb{F}_p} \frac{1}{p} \int_{\mathbb{Z}_p} \chi(\frac{a}{p^e}(b+py)^2)dy \\ &= \sum_{b\in\mathbb{F}_p} \frac{1}{p} \int_{\mathbb{Z}_p} \chi(\frac{a}{p^e}(b^2+2bpy+p^2y^2))dy \\ &= \sum_{b\in\mathbb{F}_p} \frac{1}{p}\chi(\frac{a}{p^e}(b^2)) \int_{\mathbb{Z}_p} \chi(\frac{a}{p^{e-1}}(2by+py^2))dy.\end{aligned}$$

If $b \neq 0$, we can let $u = 2by + py^2$. Then

$$du = |2b + 2py|_p dy = dy$$

since $2b + 2py$ is a p-adic unit. But

$$\chi(\frac{a}{p^e}(b^2)) \int_{\mathbb{Z}_p} \chi(\frac{a}{p^{e-1}}u)du = 0.$$

So the integral is zero unless $b = 0$, in which case the integral is 1. Then

$$\int_{\mathbb{Z}_p} \chi(\frac{a}{p^e}x^2)dx = \frac{1}{p}\int_{\mathbb{Z}_p} \chi(\frac{a}{p^e}p^2y^2)dy = \frac{1}{p}\int_{\mathbb{Z}_p} \chi(\frac{a}{p^{e-2}}y^2)dy.$$

Thus, proceeding by induction, we achieve the desired result. $\square$

Remark . *We highlight here the fact that, for* $e = 1$*, we have*

$$\int_{\mathbb{Z}_p} \chi(\frac{a}{p}x^2)dx = \sum_{b\in\mathbb{F}_p} \int_{b+p\mathbb{Z}_p} \chi(\frac{a}{p}b^2)dx = (\frac{a}{p})G.$$

This will be used repeatedly throughout the latter part of this chapter.

Next, we derive the analogous result for $p = 2$:

Lemma 3.2.2. *Assume $a \in \mathbb{Z}_2 - 2\mathbb{Z}_2$. Then*

$$\int_{\mathbb{Z}_2} \chi(\frac{a}{2^e}x^2)dx = \begin{cases} 0 & \text{if } e = 1 \\ \sqrt{2}e^{(\frac{a}{4})i\pi}2^{-\frac{e}{2}} & \text{if } \equiv 0 \ (mod\ 2), \\ \sqrt{2}e^{\{\frac{a}{8}\}i\pi}2^{-\frac{e}{2}} & \text{if } \equiv 1 \ (mod\ 2),\ e > 1. \end{cases}$$

where $\{\frac{a}{8}\}$ denotes as before the fractional part of $\frac{a}{8}$, and $(\frac{a}{4})$ is the Jacobi symbol.

Proof. For $e = 1$, this is Corollary 3.1.1. For $e = 2$,

$$\int_{\mathbb{Z}_2} \chi(\frac{a}{4}x^2)dx = \sum_{b=0}^{3} \int_{b+2^2\mathbb{Z}_2} \chi(\frac{a}{4}x^2)dx.$$

Setting $x = b + 4y$ gives $dx = \frac{1}{4}dy$, and hence the above is

$$\begin{aligned} &\frac{1}{4}\sum_{b=0}^{3} \int_{\mathbb{Z}_2} \chi(\frac{a}{4}(b+4y)^2)dy \\ =&\frac{1}{4}\sum_{b=0}^{3} \chi(\frac{a}{4}b^2) \\ =&\frac{1}{4}(2 + 2\chi(\frac{a}{4})) \\ =&\frac{1}{2}(1 + (\frac{a}{4})i) \\ =&\frac{\sqrt{2}}{2}e^{(\frac{a}{4})i\pi}. \end{aligned}$$

For $e = 3$,

$$\int_{\mathbb{Z}_2} \chi(\frac{a}{8}x^2)dx = \sum_{b=0}^{7} \int_{b+2^3\mathbb{Z}_2} \chi(\frac{a}{8}x^2)dx.$$

Setting $x = b + 8y$ gives $dx = \frac{1}{8}dy$, and hence the above is

$$\begin{aligned} &\frac{1}{8}\sum_{b=0}^{7} \int_{\mathbb{Z}_2} \chi(\frac{a}{8}(b+8y)^2)dy \\ =&\frac{1}{8}\sum_{b=0}^{7} \chi(\frac{a}{8}b^2) \\ =&\frac{1}{8}(4\chi(\frac{a}{8})) \\ =&\frac{1}{2}e^{\{\frac{a}{8}\}i\pi}. \end{aligned}$$

where the penultimate step is achieved by explicitly computing the sum.

Finally, if $e \geq 4$,

$$\begin{aligned}\int_{\mathbb{Z}_2} \chi(\frac{a}{2^e}x^2)dx &= \int_{2\mathbb{Z}_2} \chi(\frac{a}{2^e}x^2)dx + \int_{1+2\mathbb{Z}_2} \chi(\frac{a}{2^e}x^2)dx \\ &= \int_{2\mathbb{Z}_2} \chi(\frac{a}{2^e}x^2)dx + \sum_{b=1,3} \int_{b+2^2\mathbb{Z}_2} \chi(\frac{a}{2^e}x^2)dx.\end{aligned}$$

We wish to show that the sum over b is zero; this will allow us to proceed by induction. Examining only the latter integral, we set $x = b + 4y$, which gives $dx = \frac{1}{4}dy$, and hence

$$\begin{aligned}&\int_{\mathbb{Z}_2} \chi(\frac{a}{2^e}(b+4y)^2)dy \\ &\quad = \int_{\mathbb{Z}_2} \chi(\frac{a}{2^e}(b^2 + 2^3by + 2^4y^2))dy \\ &\quad = \chi(\frac{a}{2^e}b^2) \int_{\mathbb{Z}_2} \chi(\frac{a}{2^e}(2^3by + 2^4y^2))dy.\end{aligned}$$

Set $u = by + 2y^2$, which makes $du = dy$ (since b is odd). Then

$$\begin{aligned}&\int_{\mathbb{Z}_2} \chi(\frac{a}{2^e}(2^3by + 2^4y^2))dy \\ &\quad = \int_{\mathbb{Z}_2} \chi(\frac{a}{2^{e-3}}(u))du = 0\end{aligned}$$

since this is the integral of a non-trivial character over $\mathbb{Z}_2$. Thus,

$$\int_{\mathbb{Z}_2} \chi(\frac{a}{2^e}x^2)dx = \int_{2\mathbb{Z}_2} \chi(\frac{a}{2^e}x^2)dx.$$

If we change variables, letting $u = 2x$ and $du = \frac{1}{2}dx$, then

$$\int_{\mathbb{Z}_2} \chi(\frac{a}{2^e}x^2)dx = \frac{1}{2} \int_{\mathbb{Z}_2} \chi(\frac{a}{2^{e-2}}u^2)du.$$

The lemma thus follows from induction. □

3.3 Convergence

We now consider the case where $v = p$, which means that we examine solutions to $f = \nu$ over $\mathbb{Q}_p$. We must first show that, as in the real case, $|\mathcal{G}_f\varphi| \in L^1(\mathbb{Q}_p^2)$ if $n \geq 6$. We know that

$$\begin{aligned}
||\mathcal{G}_f\varphi||_1 &= \int_{\mathbb{Q}_p^2}\left|\int_{\mathbb{Z}_p^n}\chi(< f(x),\xi >)dx\right|d\xi \\
&= \int_{\mathbb{Q}_p^2}\left|\int_{\mathbb{Z}_p^n}\chi(\sum_{i=1}^n x_i^2\xi_1 + \sum_{i=1}^n \lambda_i x_i^2\xi_2)dx\right|d\xi \\
&= \int_{\mathbb{Q}_p^2}\left|\int_{\mathbb{Z}_p^n}\chi(\sum_{i=1}^n(\xi_1+\lambda_i\xi_2)x_i^2)dx\right|d\xi \\
&= \int_{\mathbb{Q}_p^2}\prod_{i=1}^n\left|\int_{\mathbb{Z}_p}\chi((\xi_1+\lambda_i\xi_2)x_i^2)dx_i\right|d\xi.
\end{aligned}$$

Now, we will use the same trick with Hölder's inequality that was used in the real case. Let us group the product into pairs, with

$$f_i = \int_{\mathbb{Z}_p^2}\chi((\xi_1+\lambda_i\xi_2)x_i^2)\chi((\xi_1+\lambda_{i+\frac{n}{2}}\xi_2)x_{i+\frac{n}{2}}^2)dx_i dx_{i+\frac{n}{2}}.$$

From Hölder's inequality, we know that

$$||\prod_{i=1}^r f_i||_1 \leq \prod_{i=1}^r ||f_i||_{p_i}.$$

We apply this to mean that

$$\begin{aligned}
&\int_{\mathbb{Q}_p^2}\prod_{i=1}^{\frac{n}{2}}\left|\int_{\mathbb{Z}_p^2}\chi((\xi_1+\lambda_i\xi_2)x_i^2)\chi((\xi_1+\lambda_{i+\frac{n}{2}}\xi_2)x_{i+\frac{n}{2}}^2)dx_i dx_{i+\frac{n}{2}}\right|d\xi \\
&\quad\leq (\prod_{i=1}^{\frac{n}{2}}\int_{\mathbb{Q}_p^2}\left|\int_{\mathbb{Z}_p^2}\chi((\xi_1+\lambda_i\xi_2)x_i^2)\chi((\xi_1+\lambda_{i+\frac{n}{2}}\xi_2)x_{i+\frac{n}{2}}^2)dx_i dx_{i+\frac{n}{2}}\right|^{\frac{n}{2}}d\xi)^{\frac{2}{n}}.
\end{aligned}$$

Now, we change variables, letting

$$z_i = \xi_1 + \lambda_i\xi_2,$$

$$w_i = \xi_1 + \lambda_{i+\frac{n}{2}}\xi_2.$$

Since the λ's are unequal, this change of variables is non-degenerate (i.e. the Jacobian is not zero). If we let J_i be the Jacobian of the transformation from (ξ_1, ξ_2) to (z_i, w_i) and let

$$J = \prod_{i=1}^{\frac{n}{2}} J_i,$$

the right side of the above inequality is then

$$(J \prod_{i=1}^{\frac{n}{2}} \int_{\mathbb{Q}_p^2} \left| \int_{\mathbb{Z}_p^2} \chi(z_i x_i^2) \chi(w_i x_{i+\frac{n}{2}}^2) dx_i dx_{i+\frac{n}{2}} \right|^{\frac{n}{2}} dz_i dw_i)^{\frac{2}{n}}.$$

Since no terms contain both z_i and w_i (or x_i and $x_{i+\frac{n}{2}}$), we can split the integrals, giving

$$\begin{aligned}(J \prod_{i=1}^{\frac{n}{2}} \int_{\mathbb{Q}_p} \left| \int_{\mathbb{Z}_p} \chi(z_i x_i^2) dx_i \right|^{\frac{n}{2}} dz_i \int_{\mathbb{Q}_p} \left| \int_{\mathbb{Z}_p} \chi(w_i x_{i+\frac{n}{2}}^2) dx_{i+\frac{n}{2}} \right|^{\frac{n}{2}} dw_i)^{\frac{2}{n}} \\ =(J \prod_{i=1}^{\frac{n}{2}} (\int_{\mathbb{Q}_p} \left| \int_{\mathbb{Z}_p} \chi(z_i x_i^2) dx_i \right|^{\frac{n}{2}} dz_i)^2)^{\frac{2}{n}}. \qquad (*)\end{aligned}$$

Now, let us evaluate

$$\int_{\mathbb{Q}_p} \left| \int_{\mathbb{Z}_p} \chi(z_i x_i^2) dx_i \right|^{\frac{n}{2}} dz_i.$$

First, we note that we can break $\mathbb{Q}_p$ into annuli as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} \left| \int_{\mathbb{Z}_p} \chi(z_i x_i^2) dx_i \right|^{\frac{n}{2}} dz_i + \sum_{e=1}^{\infty} (\int_{p^{-e}\mathbb{Z}_p - p^{-e+1}\mathbb{Z}_p} \left| \int_{\mathbb{Z}_p} \chi(z_i x_i^2) dx_i \right|^{\frac{n}{2}} dz_i) \\ = 1 + \sum_{e=1}^{\infty} \int_{p^{-e}\mathbb{Z}_p - p^{-e+1}\mathbb{Z}_p} \left| \int_{\mathbb{Z}_p} \chi(z_i x_i^2) dx_i \right|^{\frac{n}{2}} dz_i.\end{aligned}$$

We evaluate the above separately for $p \neq 2$ and $p = 2$. The two cases evaluate to similar (but not identical) expressions.

Case 1. Assume $p \neq 2$. From Lemma 3.2.1, we know that, for $z_i \in p^{-e}\mathbb{Z}_p - p^{-e+1}\mathbb{Z}_p$,

$$\begin{aligned}\left| \int_{\mathbb{Z}_p} \chi(z_i x_i^2) dx_i \right| = \begin{cases} |p^{-\frac{e}{2}}| & \text{if } e \text{ is even,} \\ |p^{-\frac{e-1}{2}} (\frac{a}{p}) G| & \text{if } e \text{ is odd,} \end{cases} \\ = p^{-\frac{e}{2}}.\end{aligned}$$

since $|G| = p^{-\frac{1}{2}}$ and $|(\frac{a}{p})| = 1$. So

$$\begin{aligned}
1+&\sum_{e=1}^{\infty}\int_{p^{-e}\mathbb{Z}_p - p^{-e+1}\mathbb{Z}_p}\left|\int_{\mathbb{Z}_p}\chi(z_i x_i^2)dx_i\right|^{\frac{n}{2}}dz_i \\
&= 1 + \sum_{e=1}^{\infty}\int_{p^{-e}\mathbb{Z}_p - p^{-e+1}\mathbb{Z}_p}|p^{-\frac{e}{2}}|^{\frac{n}{2}}dz_i \\
&= 1 + \sum_{e=1}^{\infty}p^e\frac{p-1}{p}p^{-\frac{ne}{4}} \\
&= 1 + \frac{p-1}{p}\sum_{e=1}^{\infty}p^{-\frac{(n-4)e}{4}}.
\end{aligned}$$

Case 2: Assume $p = 2$. Here, we use Lemma 3.2.2 to see that

$$\left|\int_{\mathbb{Z}_2}\chi(z_i x_i^2)dx_i\right| = \begin{cases} 0 & \text{if } e = 1 \\ |\sqrt{2}e^{(\frac{a}{4})i\pi}2^{-\frac{e}{2}}| & \text{if } \equiv 0 \ (mod\ 2), \\ |\sqrt{2}e^{\{\frac{a}{8}\}i\pi}2^{-\frac{e}{2}}| & \text{if } \equiv 1 \ (mod\ 2),\ e > 1, \end{cases}$$

$$= \begin{cases} 0 & \text{if } e = 1, \\ 2^{-\frac{e-1}{2}} & \text{if } e > 1. \end{cases}$$

So

$$\begin{aligned}
1+&\sum_{e=1}^{\infty}\int_{2^{-e}\mathbb{Z}_2 - 2^{-e+1}\mathbb{Z}_2}\left|\int_{\mathbb{Z}_2}\chi(z_i x_i^2)dx_i\right|^{\frac{n}{2}}dz_i \\
&= 1 + \sum_{e=2}^{\infty}\int_{2^{-e}\mathbb{Z}_2 - 2^{-e+1}\mathbb{Z}_2}(2^{-\frac{e-1}{2}})^{\frac{n}{2}}dz_i \\
&= 1 + \sum_{e=2}^{\infty}2^e(\frac{1}{2})2^{-\frac{n(e-1)}{4}}.
\end{aligned}$$

If we re-index, the above is

$$\begin{aligned}
1+&\sum_{e=1}^{\infty}2^{e+1}(\frac{1}{2})2^{-\frac{ne}{4}} \\
&= 1 + \sum_{e=1}^{\infty}2^{-\frac{(n-4)e}{4}}.
\end{aligned}$$

For the second case, $\frac{p-1}{p} = \frac{1}{2}$. So the first and second cases differ only by the fact that, in the second case, the infinite sum has been multiplied by 2.

Thus, returning to (∗), the above tells us that

$$\begin{aligned}
&(J\prod_{i=1}^{\frac{n}{2}}(\int_{\mathbb{Q}_p}\left|\int_{\mathbb{Z}_p}\chi(z_i x_i^2)dx_i\right|^{\frac{n}{2}}dz_i)^2)^{\frac{2}{n}} \\
&\leq (J\prod_{i=1}^{\frac{n}{2}}(1+2(\frac{p-1}{p})\sum_{e=1}^{\infty}p^{-\frac{(n-4)e}{4}})^2)^{\frac{2}{n}} \\
&= (J(1+2(\frac{p-1}{p})\sum_{e=1}^{\infty}p^{-\frac{(n-4)e}{4}})^n)^{\frac{2}{n}}.
\end{aligned}$$

Thus, the integral is finite if the sum $\sum_{e=1}^{\infty} p^{-\frac{(n-4)e}{4}}$ is convergent. For this to happen, the exponent must be negative, which occurs when

$$-\frac{(n-4)}{4} < 0,$$

or $n > 4$. Since n is even, this means that $n \geq 6$.

3.4 Evaluation of $S_p(\nu)$ for Almost All p

In this section, we will evaluate the expression $S_p(\nu)$ for all but finitely many p. This will allow us to test $S(\nu)$ for convergence; since we are only eliminating a finite set of p's, the product of the $S_p(\nu)$ over the remaining p will converge if and only the product over all p converges.

In particular, we will assume that p does not divide any of the λ's or ν's and that the λ's are all unequal. Moreover, we assume that p does not divide $\lambda_i - \lambda_j \ \forall\ i \neq j$ and $p \nmid \nu_1\lambda_i + \nu_2 \ \forall\ i$. Additionally, we assume either that

$$\frac{\nu_2}{\nu_1} \not\equiv \lambda_i \pmod{p} \ \forall\ i$$

or that

$$\frac{\nu_2}{\nu_1} = \lambda_i$$

for exactly one i. Note that the assumption of p not dividing $\lambda_i - \lambda_j \ \forall \ i, j$ means that $p \neq 2$.

Now, recall that

$$S_p(\nu) = \int_{\mathbb{Q}_p^2} \bar{\chi}(<\xi, \nu>) \int_{\mathbb{Z}_p^n} \chi(< f(x), \xi >) dx d\xi$$
$$= \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \bar{\chi}(\xi_1\nu_1 + \xi_2\nu_2) \int_{\mathbb{Z}_p^n} \chi(\sum_{i=1}^{n} (\xi_1 + \lambda_i \xi_2) x_i^2) dx d\xi_1 d\xi_2.$$

We can break up the integral over $\mathbb{Q}_p^2$ into a sum over its various annuli:

$$\int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} = \sum_{e_1=1}^{\infty} \sum_{e_2=1}^{\infty} \int_{p^{-e_1}\mathbb{Z}_p - p^{-e_1+1}\mathbb{Z}_p} \int_{p^{-e_2}\mathbb{Z}_p - p^{-e_2+1}\mathbb{Z}_p} + \sum_{e_1=1}^{\infty} \int_{p^{-e_1}\mathbb{Z}_p - p^{-e_1+1}\mathbb{Z}_p} \int_{\mathbb{Z}_p}$$
$$+ \sum_{e_2=1}^{\infty} \int_{\mathbb{Z}_p} \int_{p^{-e_1}\mathbb{Z}_p - p^{-e_1+1}\mathbb{Z}_p}.$$

The key to the evaluation of $S_p(\nu)$ will be to show the integral over any annulus is zero if either ξ_1 or ξ_2 is outside of $p^{-1}\mathbb{Z}_p$. After this, the evaluation of the integral over $\mathbb{Q}_p^2$ is merely the evaluation of the integral over $(p^{-1}\mathbb{Z}_p)^2$.

3.4.1 The Integral Outside $(p^{-1}\mathbb{Z}_p)^2$

Here, we prove the earlier claim that if the $\xi \notin (p^{-1}\mathbb{Z}_p)^2$ then the integral as ξ ranges over an annulus is zero. This will be achieved using two lemmas:

Lemma 3.4.1. *If $max\{e_1, e_2\} > 1$ and $e_1 \neq e_2$ then*

$$\int_{p^{-e_1}\mathbb{Z}_p - p^{-e_1+1}\mathbb{Z}_p} \int_{p^{-e_2}\mathbb{Z}_p - p^{-e_2+1}\mathbb{Z}_p} \bar{\chi}(\xi_1\nu_1 + \xi_2\nu_2) \int_{\mathbb{Z}_p^n} \chi(\sum_{i=1}^{n} (\xi_1 + \lambda_i \xi_2) x_i^2) dx d\xi_2 d\xi_1$$
$$= 0.$$

Proof. Let us assume first that $e_1 > e_2$. Note that

$$\begin{aligned}
&\int_{p^{-e_1}\mathbb{Z}_p - p^{-e_1+1}\mathbb{Z}_p} \int_{p^{-e_2}\mathbb{Z}_p - p^{-e_2+1}\mathbb{Z}_p} \bar{\chi}(\xi_1\nu_1 + \xi_2\nu_2) \int_{\mathbb{Z}_p^n} \chi(\sum_{i=1}^n (\xi_1 + \lambda_i\xi_2)x_i^2) dx d\xi_2 d\xi_1 \\
&\qquad = \sum_{a\in\mathbb{Z}/p\mathbb{Z}^\times} \sum_{b\in\mathbb{Z}/p\mathbb{Z}^\times} \int_{\frac{a}{p^{e_1}}+p^{-e_1+1}\mathbb{Z}_p} \int_{\frac{b}{p^{e_2}}+p^{-e_2+1}\mathbb{Z}_p} \bar{\chi}(\xi_1\nu_1 + \xi_2\nu_2) \\
&\qquad\quad \cdot \int_{\mathbb{Z}_p^n} \chi(\sum_{i=1}^n (\xi_1 + \lambda_i\xi_2)x_i^2) dx d\xi_2 d\xi_1.
\end{aligned}$$

If we let

$$\begin{aligned}
\xi_1' &= \xi_1 - \frac{a}{p^{e_1}}, \\
\xi_2' &= \xi_2 - \frac{b}{p^{e_2}},
\end{aligned}$$

a measure-invariant change of variables, then the above can be rewritten as

$$\begin{aligned}
&\sum_{a\in\mathbb{Z}/p\mathbb{Z}^\times} \sum_{b\in\mathbb{Z}/p\mathbb{Z}^\times} \int_{p^{-e_1+1}\mathbb{Z}_p} \int_{p^{-e_2+1}\mathbb{Z}_p} \bar{\chi}((\frac{a}{p^{e_1}} + \xi_1')\nu_1 + (\frac{b}{p^{e_2}} + \xi_2')\nu_2) \\
&\qquad \cdot \prod_{i=1}^n (\int_{\mathbb{Z}_p} \chi(((\frac{a}{p^{e_1}} + \xi_1') + \lambda_i(\frac{b}{p^{e_2}} + \xi_2'))x_i^2) dx_i) d\xi_2' d\xi_1' \\
&\quad = \sum_{a\in\mathbb{Z}/p\mathbb{Z}^\times} \sum_{b\in\mathbb{Z}/p\mathbb{Z}^\times} \bar{\chi}(\frac{a}{p^{e_1}}\nu_1 + \frac{b}{p^{e_2}}\nu_2) \prod_{i=1}^n A(a, e_1) \\
&\qquad \cdot \int_{p^{-e_1+1}\mathbb{Z}_p} \int_{p^{-e_2+1}\mathbb{Z}_p} \bar{\chi}(\xi_1'\nu_1 + \xi_2'\nu_2) d\xi_2' d\xi_1' \\
&\quad =(A(a, e_1))^n \sum_{a\in\mathbb{Z}/p\mathbb{Z}^\times} \sum_{b\in\mathbb{Z}/p\mathbb{Z}^\times} \bar{\chi}(\frac{a}{p^{e_1}}\nu_1 + \frac{b}{p^{e_2}}\nu_2) A_{e_1}^n \\
&\qquad \cdot \int_{p^{-e_1+1}\mathbb{Z}_p} \int_{p^{-e_2+1}\mathbb{Z}_p} \bar{\chi}(\xi_1'\nu_1 + \xi_2'\nu_2) d\xi_2' d\xi_1' \\
&\quad =(A(a, e_1))^n \sum_{a\in\mathbb{Z}/p\mathbb{Z}^\times} \sum_{b\in\mathbb{Z}/p\mathbb{Z}^\times} \bar{\chi}(\frac{a}{p^{e_1}}\nu_1 + \frac{b}{p^{e_2}}\nu_2) \int_{p^{-e_1+1}\mathbb{Z}_p} \bar{\chi}(\xi_1'\nu_1) d\xi_1' \int_{p^{-e_2+1}\mathbb{Z}_p} \bar{\chi}(\xi_2'\nu_2) d\xi_2'.
\end{aligned}$$

But since $e_1 > 1$, we know that

$$\int_{p^{-e_1+1}\mathbb{Z}_p} \bar{\chi}(\xi_1'\nu_1) d\xi_1' = 0.$$

For $e_2 > e_1$, the proof is exactly the same, except the $(A(a, e_1))^n$ is replaced with $\prod_{i=1}^n A(b\lambda_i, e_1)$. □

Lemma 3.4.2. *Let $e_1 > 1$ and $e_1 = e_2 = e$. Then*

$$\int_{p^{-e_1}\mathbb{Z}_p - p^{-e_1+1}\mathbb{Z}_p} \int_{p^{-e_2}\mathbb{Z}_p - p^{-e_2+1}\mathbb{Z}_p} \bar{\chi}(\xi_1\nu_1 + \xi_2\nu_2) \int_{\mathbb{Z}_p^n} \chi(\sum_{i=1}^{n}(\xi_1 + \lambda_i\xi_2)x_i^2) dx d\xi_2 d\xi_1$$
$$=0.$$

Proof. As before, we rewrite $p^{-e}\mathbb{Z}_p$ as a sum:

$$\begin{aligned}
&\int_{p^{-e}\mathbb{Z}_p - p^{-e+1}\mathbb{Z}_p} \int_{p^{-e}\mathbb{Z}_p - p^{-e+1}\mathbb{Z}_p} \bar{\chi}(\xi_1\nu_1 + \xi_2\nu_2) \int_{\mathbb{Z}_p^n} \chi(\sum_{i=1}^{n}(\xi_1 + \lambda_i\xi_2)x_i^2) dx d\xi_2 d\xi_1 \\
&\quad = \sum_{a\in\mathbb{Z}/p^e\mathbb{Z}^\times} \sum_{b\in\mathbb{Z}/p^e\mathbb{Z}^\times} \int_{\frac{a}{p^e}+p^{-e+1}\mathbb{Z}_p} \int_{\frac{b}{p^e}+p^{-e+1}\mathbb{Z}_p} \bar{\chi}(\xi_1\nu_1 + \xi_2\nu_2) \\
&\qquad \cdot \int_{\mathbb{Z}_p^n} \chi(\sum_{i=1}^{n}(\xi_1 + \lambda_i\xi_2)x_i^2) dx d\xi_2 d\xi_1.
\end{aligned}$$

We change variables as in the previous lemma, making the above expression into

$$\begin{aligned}
&\sum_{a\in\mathbb{Z}/p^e\mathbb{Z}^\times} \sum_{b\in\mathbb{Z}/p^e\mathbb{Z}^\times} \int_{p^{-e+1}\mathbb{Z}_p} \int_{p^{-e+1}\mathbb{Z}_p} \bar{\chi}((\frac{a}{p^e} + \xi_1')\nu_1 + (\frac{b}{p^e} + \xi_2')\nu_2) \\
&\qquad \cdot \prod_{i=1}^{n}(\int_{\mathbb{Z}_p} \chi(((\frac{a}{p^e} + \xi_1') + \lambda_i(\frac{b}{p^e} + \xi_2'))x_i^2) dx_i) d\xi_2' d\xi_1' \\
&\quad = \sum_{a\in\mathbb{Z}/p^e\mathbb{Z}^\times} \sum_{b\in\mathbb{Z}/p^e\mathbb{Z}^\times} \int_{p^{-e+1}\mathbb{Z}_p} \int_{p^{-e+1}\mathbb{Z}_p} \bar{\chi}(\frac{a}{p^e}\nu_1 + \frac{b}{p^e}\nu_2) \\
&\qquad \cdot (\xi_1'\nu_1 + \xi_2'\nu_2) \prod_{i=1}^{n}(\int_{\mathbb{Z}_p} \chi(((\frac{a}{p^e} + \lambda_i\frac{b}{p^e}) + (\xi_1' + \lambda_i\xi_2'))x_i^2) dx_i) d\xi_2' d\xi_1'.
\end{aligned}$$

Now, for a given a and b, if $\frac{a}{p^e} + \lambda_j\frac{b}{p^e} \notin p^{-e}\mathbb{Z}_p - p^{-e+1}\mathbb{Z}_p$ then

$$a + \lambda_j b \equiv 0 \pmod{p}.$$

But then

$$a + \lambda_i b \not\equiv 0 \pmod{p}\ \forall\ i \neq j$$

else $\lambda_j \equiv \lambda_i$, contradicting our assumption that $p \nmid \lambda_j - \lambda_i$. So, for all but at most one λ (which we denote λ_j),

$$\frac{a}{p^e} + \lambda_i\frac{b}{p^e} \in p^{-e}\mathbb{Z}_p - p^{-e+1}\mathbb{Z}_p.$$

This means that, for these $i \neq j$,

$$\int_{\mathbb{Z}_p} \chi((\frac{a}{p^e} + \lambda_i \frac{b}{p^e})x_i^2) = A(a + \lambda_i b, e).$$

So let λ_j be as above. If no such λ exists, choose any of the λ's to be λ_j. Then our integral expression is

$$\sum_{a \in \mathbb{Z}/p^e\mathbb{Z}^\times} \sum_{b \in \mathbb{Z}/p^e\mathbb{Z}^\times} \prod_{1 \leq i \leq n, i \neq j} A(a + \lambda_i b, e)$$
$$\cdot \int_{p^{-e+1}\mathbb{Z}_p} \int_{p^{-e+1}\mathbb{Z}_p} \bar{\chi}(\frac{a}{p^e}\nu_1 + \frac{b}{p^e}\nu_2)\bar{\chi}(\xi_1'\nu_1 + \xi_2'\nu_2)$$
$$\cdot (\int_{\mathbb{Z}_p} \chi(((\frac{a}{p^e} + \lambda_j \frac{b}{p^e}) + (\xi_1' + \lambda_i \xi_2'))x_j^2) dx_j) d\xi_2' d\xi_1'.$$

If we let

$$z = \xi_1'\nu_1 + \xi_2'\nu_2,$$

$$w = \xi_1' + \lambda_j \xi_2'$$

then the change of measure is

$$|\nu_1\lambda_j + \nu_2|_p^{-1}.$$

By assumption, $|\nu_1\lambda_j + \nu_2|_p^{-1}$ is non-zero. Thus, the integral is

$$|\nu_1\lambda_j + \nu_2|_p^{-1} \prod_{1 \leq i \leq n, i \neq j} A(a + \lambda_i b, e) \int_{p^{-e+1}\mathbb{Z}_p} \int_{p^{-e+1}\mathbb{Z}_p} \bar{\chi}(\frac{a}{p^e}\nu_1 + \frac{b}{p^e}\nu_2)\bar{\chi}(z)$$
$$\cdot (\int_{\mathbb{Z}_p} \chi((\frac{a}{p^e} + \lambda_j \frac{b}{p^e} + w)x_j^2) dx_j) dw dz.$$

Since $e > 1$, we know that

$$\int_{p^{-e+1}\mathbb{Z}_p} \bar{\chi}(z) = 0.$$

Thus the lemma holds. $\square$

3.4.2 The Integral Over $(p^{-1}\mathbb{Z}_p)^2$

Now that we have determined that the $S_p(\nu)$ is entirely determined by the annuli where $e_i \leq 1$, we will calculate the integral over each of the remaining annuli individually.

Lemma 3.4.3. *Let $e_1 = e_2 = 1$. Assume that n is even and*

$$\frac{\nu_2}{\nu_1} \not\equiv \lambda_i \ \forall \ i.$$

(Here, all congruences are modulo p.) Additionally, define

$$H = \{u \in \mathbb{F}_p^\times : u \not\equiv -\lambda_i^{-1} \ \forall \ i, \ u \not\equiv -\nu_1\nu_2^{-1}\},$$

and let G be the classical Gauss sum as defined before. Then

$$\int_{p^{-1}\mathbb{Z}_p-\mathbb{Z}_p}\int_{p^{-1}\mathbb{Z}_p-\mathbb{Z}_p}\bar{\chi}(\xi_1\nu_1+\xi_2\nu_2)\int_{\mathbb{Z}_p^n}\chi(\sum_{i=1}^n(\xi_1+\lambda_i\xi_2)x_i^2)dx d\xi_2 d\xi_1$$
$$= -\sum_{u\in H}\prod_{i=1}^n(\frac{1+\lambda_i u}{p})G^n - G^n p\sum_{j=1}^n\prod_{1\leq i\leq n, i\neq j}(\frac{1+\lambda_i-\lambda_j^{-1}}{p})(\frac{-\nu_1+\lambda_j^{-1}\nu_2}{p})$$
$$+(p-1)\prod_{i=1}^n(\frac{1-\lambda_i\nu_1\nu_2^{-1}}{p})G^n.$$

Proof. First, we rewrite the above integral as

$$\sum_{a\in\mathbb{F}_p^\times}\sum_{b\in\mathbb{F}_p^\times}\int_{\frac{a}{p}+\mathbb{Z}_p}\int_{\frac{b}{p}+\mathbb{Z}_p}\bar{\chi}(\frac{a}{p}\nu_1+\frac{b}{p}\nu_2)(\prod_{i=1}^n\int_{\mathbb{Z}_p}\chi(\frac{a+\lambda_i b}{p}x_i^2)dx_i)d\xi_1 d\xi_2$$
$$= \sum_{a\in\mathbb{F}_p^\times}\sum_{b\in\mathbb{F}_p^\times}\bar{\chi}(\frac{a}{p}\nu_1+\frac{b}{p}\nu_2)(\prod_{i=1}^n\int_{\mathbb{Z}_p}\chi(\frac{a+\lambda_i b}{p}x_i^2)dx_i).$$

Now, since $a, b \in \mathbb{F}_p^\times$, we can write $b = au$ for some $u \in \mathbb{F}_p^\times$. So the above can instead be written as

$$\sum_{a\in\mathbb{F}_p^\times}\sum_{u\in\mathbb{F}_p^\times}\bar{\chi}(\frac{a}{p}(\nu_1+u\nu_2))(\prod_{i=1}^n\int_{\mathbb{Z}_p}\chi(\frac{a(1+\lambda_i u)}{p}x_i^2)dx_i)$$
$$= \sum_{u\in\mathbb{F}_p^\times}\sum_{a\in\mathbb{F}_p^\times}\bar{\chi}(\frac{a}{p}(\nu_1+u\nu_2))(\prod_{i=1}^n\int_{\mathbb{Z}_p}\chi(\frac{a(1+\lambda_i u)}{p}x_i^2)dx_i).$$

Now, since the λ's are incongruent modulo p, we can break the u's into three sets. These sets are

$$H = \{u \in \mathbb{F}_p^\times : u \not\equiv -\lambda_i^{-1} \ \forall\ i,\ u \not\equiv -\nu_1\nu_2^{-1}\},$$

$$B = \{u \in \mathbb{F}_p^\times : u \equiv -\lambda_i^{-1} \text{ for some } i,\ u \not\equiv -\nu_1\nu_2^{-1}\},$$

$$D = \{u \equiv -\nu_1\nu_2^{-1}\}.$$

We deal first with $u \in H$. Then $1 + \lambda_i u \not\equiv 0$. Here,

$$\begin{aligned}\int_{\mathbb{Z}_p} \chi(\frac{a(1+\lambda_i u)}{p} x_i^2) dx_i &= (\frac{a(1+\lambda_i u)}{p})G \\ &= (\frac{a}{p})(\frac{(1+\lambda_i u)}{p})G.\end{aligned}$$

So

$$\begin{aligned}&\sum_{u \in H} \sum_{a \in \mathbb{F}_p^\times} \bar{\chi}(\frac{a}{p}(\nu_1 + u\nu_2))(\prod_{i=1}^{n} \int_{\mathbb{Z}_p} \chi(\frac{a(1+\lambda_i u)}{p} x_i^2) dx_i) \\ &= G^n \sum_{u \in H} \sum_{a \in \mathbb{F}_p^\times} \bar{\chi}(\frac{a}{p}(\nu_1 + u\nu_2)) \prod_{i=1}^{n} (\frac{a}{p})(\frac{1+\lambda_i u}{p}) \\ &= G^n \sum_{u \in H} \sum_{a \in \mathbb{F}_p^\times} (\frac{a}{p})^n \bar{\chi}(\frac{a}{p}(\nu_1 + u\nu_2)) \prod_{i=1}^{n} (\frac{1+\lambda_i u}{p}).\end{aligned}$$

If n is even, the above is

$$\begin{aligned}G^n &\sum_{u \in H} \sum_{a \in \mathbb{F}_p^\times} \bar{\chi}(\frac{a}{p}(\nu_1 + u\nu_2)) \prod_{i=1}^{n} (\frac{1+\lambda_i u}{p}) \\ &= G^n \sum_{u \in H} \prod_{i=1}^{n} (\frac{1+\lambda_i u}{p}) \sum_{a \in \mathbb{F}_p^\times} \bar{\chi}(\frac{a}{p}(\nu_1 + u\nu_2)) \\ &= G^n \sum_{u \in H} \prod_{i=1}^{n} (\frac{1+\lambda_i u}{p})(-1).\end{aligned}$$

since the sum of the a's is the sum of all of the p-th roots of unity except 1. This gives us the first summand from the right side of the equality in the statement of the lemma.

Next, we deal with the case of $u \in B$. In this case, $\exists\ i$ such that $1 + u\lambda_j \equiv 0$. So, for this j,

$$\int_{\mathbb{Z}_p} \chi(\frac{a(1+\lambda_j u)}{p} x_j^2) dx_i = 1.$$

Then, $\forall\ i \neq j$,

$$\int_{\mathbb{Z}_p} \chi(\frac{a(1+\lambda_i u)}{p} x_i^2) dx_i = (\frac{a}{p})(\frac{(1+\lambda_i u)}{p})G$$

as before. So

$$\begin{aligned}
&\sum_{u\in B}\sum_{a\in\mathbb{F}_p^\times} \bar{\chi}(\frac{a}{p}(\nu_1+u\nu_2))(\prod_{i=1}^n \int_{\mathbb{Z}_p} \chi(\frac{a(1+\lambda_i u)}{p} x_i^2) dx_i) \\
&\quad = G^{n-1} \sum_{u\in B} \prod_{1\le i\le n, i\neq j} (\frac{1+\lambda_i u}{p}) \sum_{a\in\mathbb{F}_p^\times} \bar{\chi}(\frac{a}{p}(\nu_1+u\nu_2))(\frac{a}{p})^{n-1} \\
&\quad = G^{n-1} \sum_{u\in B} \prod_{1\le i\le n, i\neq j} (\frac{1+\lambda_i u}{p}) \sum_{a\in\mathbb{F}_p^\times} \bar{\chi}(\frac{a}{p}(\nu_1+u\nu_2))(\frac{a}{p}).
\end{aligned}$$

Now, from Theorem 3.1.2, we know that if $p \nmid r$ then

$$\sum_{x\in\mathbb{F}_p} (\frac{x}{p})\chi(rx) = (\frac{r}{p}) \sum_{x\in\mathbb{F}_p} (\frac{rx}{p})\chi(rx) = (\frac{r}{p}) \sum_{x=0}^{p-1} \chi(x^2) = (\frac{r}{p})Gp.$$

In our case,

$$\sum_{a\in\mathbb{F}_p^\times} \bar{\chi}(\frac{a}{p}(\nu_1+u\nu_2))(\frac{a}{p}) = (\frac{-(\nu_1+u\nu_2)}{p})Gp.$$

So the above is

$$G^n p \sum_{u\in B} \prod_{1\le i\le n, i\neq j} (\frac{1+\lambda_i u}{p})(\frac{-(\nu_1+u\nu_2)}{p}).$$

Since $u \in B$ means $u \equiv -\lambda_j^{-1}$ for some j, the above is

$$G^n p \sum_{j=1}^n \prod_{1\le i\le n, i\neq j} (\frac{1+\lambda_i - \lambda_j^{-1}}{p})(\frac{-\nu_1+\lambda_j^{-1}\nu_2}{p}).$$

This is the second summand in the lemma.

Finally, we deal with the case where $u \equiv -\nu_1\nu_2^{-1}$. In this case, we know that

$1 + u\lambda_i \not\equiv 0 \; \forall \; i$. So

$$\int_{\mathbb{Z}_p} \chi(\frac{a(1+\lambda_i u)}{p} x_i^2) dx_i = (\frac{a(1+\lambda_i u)}{p}) G \; \forall \; i.$$

and hence

$$\begin{aligned}
&\sum_{u \in D} \sum_{a \in \mathbb{F}_p^\times} \bar{\chi}(\frac{a}{p}(\nu_1 + u\nu_2))(\prod_{i=1}^n \int_{\mathbb{Z}_p} \chi(\frac{a(1+\lambda_i u)}{p} x_i^2) dx_i) \\
&= G^n \sum_{u \in D} \prod_{i=1}^n (\frac{1+\lambda_i u}{p}) \sum_{a \in \mathbb{F}_p^\times} \bar{\chi}(\frac{a}{p}(\nu_1 + u\nu_2))(\frac{a}{p})^n \\
&= G^n \prod_{i=1}^n (\frac{1-\lambda_i \nu_1 \nu_2^{-1}}{p}) \sum_{a \in \mathbb{F}_p^\times} \bar{\chi}(0) \\
&= (p-1) G^n \prod_{i=1}^n (\frac{1-\lambda_i \nu_1 \nu_2^{-1}}{p}).
\end{aligned}$$

This is the third summand in the lemma. □

Lemma 3.4.3.1. *Let $e_1 = e_2 = 1$. We make the same assumptions and definitions as in Lemma 3.4.3, except that*

$$\frac{\nu_2}{\nu_1} = \lambda_k$$

for exactly one k. Then

$$\begin{aligned}
&\int_{p^{-1}\mathbb{Z}_p - \mathbb{Z}_p} \int_{p^{-1}\mathbb{Z}_p - \mathbb{Z}_p} \bar{\chi}(\xi_1 \nu_1 + \xi_2 \nu_2) \int_{\mathbb{Z}_p^n} \chi(\sum_{i=1}^n (\xi_1 + \lambda_i \xi_2) x_i^2) dx d\xi_2 d\xi_1 \\
&\quad = -\sum_{u \in H} \prod_{i=1}^n (\frac{1+\lambda_i u}{p}) G^n - G^n p \sum_{1 \le j \le n, j \ne k} \prod_{1 \le i \le n, i \ne j} (\frac{1+\lambda_i - \lambda_j^{-1}}{p})(\frac{-\nu_1 + \lambda_j^{-1}\nu_2}{p}) \\
&\qquad + (p-1) \prod_{1 \le i \le n, i \ne k} (\frac{1-\lambda_i \nu_1 \nu_2^{-1}}{p}) G^{n-1}.
\end{aligned}$$

Proof. : The proof for the first term (i.e. for $u \in H$) is exactly the same. For the second term (where $u \in B$), we note that B does not include $-\lambda_k^{-1}$ since $-\lambda_k^{-1} = \nu_1 \nu_2^{-1}$. For the term where $u \in D$, we note that

$$\int_{\mathbb{Z}_p} \chi(\frac{a(1+\lambda_i u)}{p} x_k^2) dx_k) = 1.$$

So

$$\sum_{u\in D}\sum_{a\in\mathbb{F}_p^\times}\bar{\chi}(\frac{a}{p}(\nu_1+u\nu_2))(\prod_{i=1}^n\int_{\mathbb{Z}_p}\chi(\frac{a(1+\lambda_i u)}{p}x_i^2)dx_i)$$

$$=G^{n-1}\sum_{u\in D}\prod_{1\le i\le n, i\neq k}(\frac{1+\lambda_i u}{p})\sum_{a\in\mathbb{F}_p^\times}\bar{\chi}(\frac{a}{p}(\nu_1+u\nu_2))(\frac{a}{p})^n$$

$$=G^{n-1}\prod_{i=1}^n(\frac{1-\lambda_i\nu_1\nu_2^{-1}}{p})\sum_{a\in\mathbb{F}_p^\times}\bar{\chi}(0)$$

$$=(p-1)G^n\prod_{1\le i\le n, i\neq k}(\frac{1-\lambda_i\nu_1\nu_2^{-1}}{p}).$$

□

Lemma 3.4.4. *Let $e_1=1$, $e_2=0$. Then*

$$\int_{p^{-1}\mathbb{Z}_p-\mathbb{Z}_p}\int_{\mathbb{Z}_p}\bar{\chi}(\xi_1\nu_1+\xi_2\nu_2)\int_{\mathbb{Z}_p^n}\chi(\sum_{i=1}^n(\xi_1+\lambda_i\xi_2)x_i^2)dx d\xi_2 d\xi_1$$

$$=\begin{cases}-G^n & \text{if } n \text{ is even,}\\ G^{n+1}(\frac{-\nu_1}{p}) & \text{if } n \text{ is odd.}\end{cases}$$

Proof. In this case, the above integral is rewritten as

$$\int_{p^{-1}\mathbb{Z}_p-\mathbb{Z}_p}\int_{\mathbb{Z}_p}\bar{\chi}(\xi_1\nu_1)\int_{\mathbb{Z}_p^n}\chi(\sum_{i=1}^n(\xi_1)x_i^2)dx d\xi_2 d\xi_1$$

$$=\sum_{a\in\mathbb{F}_p^\times}\int_{\frac{a}{p}+\mathbb{Z}_p}\int_{\mathbb{Z}_p}\bar{\chi}(\frac{a}{p}\nu_1)\int_{\mathbb{Z}_p}\chi(\sum_{i=1}^n\frac{a}{p}\cdot x_i^2)dx d\xi_2 d\xi_1$$

$$=\sum_{a\in\mathbb{F}_p^\times}\int_{\frac{a}{p}+\mathbb{Z}_p}\int_{\mathbb{Z}_p}\bar{\chi}(\frac{a}{p}\nu_1)(\int_{\mathbb{Z}_p}\chi(\frac{a}{p}\cdot x_i^2)dx_i)^n d\xi_2 d\xi_1$$

$$=\sum_{a\in\mathbb{F}_p^\times}\bar{\chi}(\frac{a}{p}\nu_1)((\frac{a}{p})G)^n.$$

If n is even, the above equals

$$G^n\sum_{a\in\mathbb{F}_p^\times}\bar{\chi}(\frac{a}{p}\nu_1)$$

$$=-G^n.$$

If n is odd, the integral expression instead equals

$$\sum_{a\in\mathbb{F}_p^\times}\bar{\chi}(\frac{a}{p}\nu_1)(\frac{a}{p})G^n$$

$$= G^n\sum_{a\in\mathbb{F}_p^\times}\bar{\chi}(\frac{a^2}{p}\nu_1)(\frac{a}{p})$$

$$= G^{n+1}(\frac{-\nu_1}{p}).$$

□

Lemma 3.4.5. *Let $e_1 = 0$, $e_2 = 1$. Then*

$$\int_{\mathbb{Z}_p}\int_{p^{-1}\mathbb{Z}_p-\mathbb{Z}_p}\bar{\chi}(\xi_1\nu_1+\xi_2\nu_2)\int_{\mathbb{Z}_p^n}\chi(\sum_{i=1}^n(\xi_1+\lambda_i\xi_2)x_i^2)dxd\xi_2 d\xi_1$$

$$=\begin{cases}-(\prod_{i=1}^n\frac{\lambda_i}{p})G^n & \text{if } n \text{ is even,}\\ (\prod_{i=1}^n\frac{\lambda_i}{p})G^{n+1}(\frac{-\nu_2}{p}) & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Here, we note that the above integral is equal to

$$\sum_{b\in\mathbb{F}_p^\times}\int_{\mathbb{Z}_p}\int_{\frac{b}{p}+\mathbb{Z}_p}\bar{\chi}(\frac{b}{p}\nu_2)\int_{\mathbb{Z}_p^n}\chi(\sum_{i=1}^n(\lambda_i\frac{b}{p})x_i^2)dxd\xi_2 d\xi_1$$

$$=\sum_{b\in\mathbb{F}_p^\times}\bar{\chi}(\frac{b}{p}\nu_2)\prod_{i=1}^n\int_{\mathbb{Z}_p}\chi((\lambda_i\frac{b}{p})x_i^2)dx_i$$

$$=\sum_{b\in\mathbb{F}_p^\times}\bar{\chi}(\frac{b}{p}\nu_2)(\prod_{i=1}^n(\frac{\lambda_i b}{p})G)$$

$$=G^n\prod_{i=1}^n(\frac{\lambda_i}{p})\sum_{b\in\mathbb{F}_p^\times}\bar{\chi}(\frac{b}{p}\nu_2)((\frac{b}{p}))^n.$$

If n is even then

$$\sum_{b\in\mathbb{F}_p^\times}\bar{\chi}(\frac{b}{p}\nu_2)((\frac{b}{p}))^n=\sum_{b\in\mathbb{F}_p^\times}\bar{\chi}(\frac{b}{p}\nu_2)=-1.$$

If n is odd then

$$\sum_{b\in\mathbb{F}_p^\times}\bar{\chi}(\frac{b}{p}\nu_2)((\frac{b}{p}))^n=\sum_{b\in\mathbb{F}_p^\times}\bar{\chi}(\frac{b}{p}\nu_2)(\frac{b}{p})=(\frac{-\nu_2}{p})G.$$

Thus, the lemma follows. □

Lemma 3.4.6. *Let $e_1 = e_2 = 0$. Then*

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\bar{\chi}(\xi_1\nu_1+\xi_2\nu_2)\int_{\mathbb{Z}_p^n}\chi(\sum_{i=1}^{n}(\xi_1+\lambda_i\xi_2)x_i^2)dxd\xi_2d\xi_1 = 1.$$

Proof. If $e_1 = e_2 = 0$ then $\xi \in \mathbb{Z}_p^2$. Since $\nu_i, x_i \in \mathbb{Z}_p$ and $\chi(\mathbb{Z}_p) = 1$, it follows that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\bar{\chi}(\xi_1\nu_1+\xi_2\nu_2)\int_{\mathbb{Z}_p^n}\chi(\sum_{i=1}^{n}(\xi_1+\lambda_i\xi_2)x_i^2)dxd\xi_2d\xi_1\\
&= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p^n}dxd\xi_2d\xi_1\\
&= 1.
\end{aligned}$$

□

Finally, we arrive at the main theorems of this section. From the above lemmas, we have proven the following:

Theorem 3.4.7. *Let $p \nmid \lambda_i, \nu_1, \nu_2$, $p \nmid \lambda_i - \lambda_j$, $p \nmid \nu_1\lambda_1 + \nu_2$. Moreover, as above, let*

$$H = \{u \in \mathbb{F}_p^\times : u \not\equiv -\lambda_i^{-1}\ \forall\ i,\ u \not\equiv -\nu_1\nu_2^{-1}\}.$$

Assume that n is even and

$$\lambda_i \not\equiv \frac{\nu_2}{\nu_1}.$$

Then

$$\begin{aligned}
S_p(\nu) = &-\sum_{u\in H}\prod_{i=1}^{n}(\frac{1+\lambda_i u}{p})G^n - G^n p\sum_{j=1}^{n}\prod_{1\le i\le n, i\ne j}(\frac{1+\lambda_i-\lambda_j^{-1}}{p})(\frac{-\nu_1+\lambda_j^{-1}\nu_2}{p})\\
&+(p-1)\prod_{i=1}^{n}(\frac{1-\lambda_i\nu_1\nu_2^{-1}}{p})G^n - G^n - G^n(\prod_{i=1}^{n}(\frac{\lambda_i}{p}))+1.
\end{aligned}$$

Proof. If we make the same assumptions as in Theorem 3.4.1 except

$$\lambda_k = \frac{\nu_2}{\nu_1},$$

then

$$S_p(\nu) = -\sum_{u\in H}\prod_{i=1}^{n}(\frac{1+\lambda_i u}{p})G^n - G^n p\sum_{1\le j\le n, j\ne k}\prod_{1\le i\le n, i\ne j}(\frac{1+\lambda_i-\lambda_j^{-1}}{p})(\frac{-\nu_1+\lambda_j^{-1}\nu_2}{p})$$
$$+(p-1)\prod_{1\le i\le n, i\ne k}(\frac{1-\lambda_i\nu_1\nu_2^{-1}}{p})G^{n-1} - G^n - G^n(\prod_{i=1}^{n}(\frac{\lambda_i}{p}))+1.$$

□

3.5 Convergence of the Singular Series

Here, we will use the computations of the previous section to determine whether our expression for $S(\nu)$ converges. Recall that the singular series $S(\nu)$ can be found by

$$S(\nu) = \prod_p S_p(\nu)$$

where S_p are the local singular series as above. We prove the following theorem:

Theorem 3.5.1. *If $\lambda_k \ne \frac{\nu_2}{\nu_1}$ and $n \ge 6$ then $S(\nu)$ converges.*

Proof. First, from Theorem 3.4.7, the product

$$T(\nu) = \prod_{p\ prime}(-\sum_{u\in H}\prod_{i=1}^{n}(\frac{1+\lambda_i u}{p})G^n - G^n p\sum_{j=1}^{n}\prod_{1\le i\le n, i\ne j}(\frac{1+\lambda_i-\lambda_j^{-1}}{p})(\frac{-\nu_1+\lambda_j^{-1}\nu_2}{p}))$$
$$+(p-1)\prod_{i=1}^{n}(\frac{1-\lambda_i\nu_1\nu_2^{-1}}{p})G^n - G^n - G^n(\prod_{i=1}^{n}(\frac{\lambda_i}{p}))+1$$

differs from the product

$$S(\nu) = \prod_p S_p(\nu)$$

by only a finite number of terms. So if we can show that $T(\nu)$ converges then $S(\nu)$

must also converge. So

$$
\begin{aligned}
|T(\nu)| \leq \prod_{p\ prime} \sum_{u \in H} \prod_{i=1}^{n} |(\frac{1+\lambda_i u}{p})G^n| + |G^n p| \sum_{j=1}^{n} \prod_{1\leq i\leq n, i\neq j} |(\frac{1+\lambda_i - \lambda_j^{-1}}{p})(\frac{-\nu_1+\lambda_j^{-1}\nu_2}{p})| \\
+ (p-1)\prod_{i=1}^{n} |(\frac{1-\lambda_i\nu_1\nu_2^{-1}}{p})G^n| + |G^n| + |G^n||(\prod_{i=1}^{n}(\frac{\lambda_i}{p}))| + 1 \\
\leq \prod_{p\ prime} \sum_{u\in H} p^{-\frac{n}{2}} + p^{1-\frac{n}{2}}(\sum_{j=1}^{n} 1) + (p-1)p^{-\frac{n}{2}} + p^{-\frac{n}{2}} + p^{-\frac{n}{2}} + 1 \\
= \prod_{p\ prime} (p-n-1)p^{-\frac{n}{2}} + np^{-\frac{n-2}{2}} + (p+1)p^{-\frac{n}{2}} + 1 \\
= \prod_{p\ prime} -np^{-\frac{n}{2}} + (n+2)p^{-\frac{n-2}{2}} + 1.
\end{aligned}
$$

Now, we use the fact that the product $\prod_{i=1}^{\infty}(1+a_i)$ converges if and only if the sum $\sum_{i=1}^{\infty} a_i$ does as well. Then $T(\nu)$ converges if

$$
\begin{aligned}
\sum_{i=1}^{\infty} - ni^{-\frac{n}{2}} + (n+2)i^{-\frac{n-2}{2}} \\
= -n\sum_{i=1}^{\infty} i^{-\frac{n}{2}} + (n+2)\sum_{i=1}^{\infty} i^{-\frac{n-2}{2}}
\end{aligned}
$$

converges. We know that a sum like $\sum_{i=1}^{\infty} i^k$ converges if $k > 1$. So $\sum_{i=1}^{\infty} i^{-\frac{n}{2}}$ converges if $n > 2$, and $\sum_{i=1}^{\infty} i^{-\frac{n-2}{2}}$ converges if $n > 4$. Since we assumed that n is even, $n \geq 6$ is required for both summations to converge. □

Theorem 3.5.2. *If $\lambda_k = \frac{\nu_2}{\nu_1}$ for one k and $n \geq 6$ then $S(\nu)$ converges.*

Proof. Again, we consider $T(\nu)$ and note that it differs from $S(\nu)$ by only a finite number of terms. In this case,

$$
\begin{aligned}
|T(\nu)| \leq \sum_{u\in H} |\prod_{i=1}^{n} |(\frac{1+\lambda_i u}{p})||G^n| + |G^n|p \sum_{1\leq j\leq n, j\neq k} \prod_{1\leq i\leq n, i\neq j} |(\frac{1+\lambda_i-\lambda_j^{-1}}{p})(\frac{-\nu_1+\lambda_j^{-1}\nu_2}{p})| \\
+ (p-1)\prod_{1\leq i\leq n, i\neq k} |(\frac{1-\lambda_i\nu_1\nu_2^{-1}}{p})G^{n-1}| + |G^n| + |G^n(\prod_{i=1}^{n}(\frac{\lambda_i}{p}))| + 1 \\
\leq \prod_{p\ prime} (p-n)p^{-\frac{n}{2}} + (n-1)p^{1-\frac{n}{2}} + (p-1)p^{-\frac{n-1}{2}} + p^{-\frac{n}{2}} + p^{-\frac{n}{2}} + 1 \\
= \prod_{p\ prime} p^{-\frac{n-3}{2}} + np^{-\frac{n-2}{2}} - p^{-\frac{n-1}{2}} + (2-n)p^{-\frac{n}{2}} + 1.
\end{aligned}
$$

As before, this converges if and only if

$$\sum_{p\ prime} p^{-\frac{n-4}{2}} + np^{-\frac{n-2}{2}} - p^{-\frac{n-1}{2}} + (2-n)p^{-\frac{n}{2}}$$

$$= \sum_{p\ prime} p^{-\frac{n-3}{2}} + \sum_{p\ prime} np^{-\frac{n-2}{2}} - \sum_{p\ prime} p^{-\frac{n-1}{2}}$$

$$+ \sum_{p\ prime} (2-n)p^{-\frac{n}{2}}$$

converges. But these four sums all converge for $n \geq 6$. □

Bibliography

[1] F. Bowman. *Introduction to Bessel Functions*. Dover Publications, Inc., New York, 1958.

[2] R. N. Bracewell. *The Fourier Transform and Its Applications*. McGraw-Hill Series in Electrical Engineering. Circuits and Systems. McGraw-Hill Book Co., New York, 2000. Third Edition.

[3] G. H. Hardy and J. E. Littlewood. A new solution to waring's problem. *Q. J. Math.*, 48:272–293, 1919.

[4] G. Lachaud. Une présentation adélique de la série singuliére et du probléme de waring. *Enseign. Math.*, 28:139–169, 1982.

[5] T. Ono. Gauss transforms and zeta-functions. *Ann. of Math.*, 91:332–361, 1970.

[6] T. Ono. *Lectures on the Hardy-Littlewood singular series*. Transcribed by T. Wright, 2008.

[7] C. L. Siegel. *Analytische Zahlentheorie : Vorlesung[en] gehalten im Sommersemester 1963 [und im Wintersemester 1963/64] an der Universitt Gttingen*. Mathematisches Institut der Universität Göttingen, Göttingen, 1964.

Vita

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