STRONG RATIONAL CONNECTEDNESS OF TORIC VARIETIES

by

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ABSTRACT

In this thesis, we discuss the following problem:

Let X be an FT (Fano type) variety, and $P_1, \ldots, P_r \in X$ be finitely many points (possibly singular). Is there a geometrically free rational curve $f : \mathbb{P}^1 \to X$ over $P_1, \ldots, P_r$?

The answer is 'yes' when X is a projective toric variety over $\mathbb{C}$. As a corollary, we shall prove that the smooth loci of projective toric varieties over $\mathbb{C}$ is strongly rationally connected.

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1 Introduction and Preliminaries

1.1 Introduction

In recent years, there have been two main streams of development in birational geometry. The first breakthrough comes from Hacon, McKernan and their coauthors. Using ideas from Siu, Kawamata and Shokurov, they announced a series of fundamental results of the minimal model program (MMP): the finite generation of canonical rings and the existence of minimal models of varieties of general type ([BCHM06]). The second important advance comes from the study of rationally connected varieties, a concept independently invented by Kollár-Miyaoka-Mori ([KMM92b]) and Campana([Ca92]). This kind of variety has interesting arithmetic and geometric properties ([Kol96],[KMM92b]) and is considered quite fundamental ([Kol01]).

These two main streams are related. It has become increasingly clear that the birational geometry of higher dimensional varieties is closely related to the geometry of rational curves on those varieties. From the point of view of the minimal model program, one expects that rational curves on varieties with mild singularities (e.g., log terminal singularities) share many of the basic properties of rational curves on smooth varieties. Surprisingly, very little seems to be known in this direction.

Roughly speaking, a variety X is called rationally connected if for any two generic points $x_1, x_2 \in X$, there is an irreducible rational curve connecting x_1 and x_2, see Definition 3.1.1.

A class of proper rationally connected varieties comes from the smooth Fano varieties ([Ca92], [KMM92a] or [Kol96]). Shokurov ([Sh00]) proved that FT (Fano type) varieties (see Definition 1.2.7) are rationally connected varieties assuming LMMP

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(Log Minimal Model Program). Q. Zhang ([Zh06]), Hacon and McKernan ([HM07]) proved the same result without assuming LMMP. The condition of klt singularities cannot be weakened to log canonical singularities (see 2.2 in [KMM92b]). However the rational chain connectedness holds for the log canonical Fano varieties [Sh00].

A harder question is whether the smooth locus of a rationally connected variety is rationally connected. In general the rational connectedness of a nonproper variety is subtle. D. Zhang ([Zh95]) gave an example of a projective rational surface S with only rational double point singularities, but its smooth locus is not rationally connected (see Example 3.2.15). This example is not FT. Now people ask whether the smooth loci of FT varieties are rationally connected. For the surface case, Keel and McKernan gave an affirmative answer, that is, if (S, Δ) is a log del Pezzo surface, then its smooth locus S^{sm} is rationally connected ([KM99]), but this does not imply the strong rational connectedness.

The concept of strongly rationally connected varieties (see Definition 3.2.14) was first introduced by Hassett and Tschinkel, see Definition 3.2.14. A proper and smooth rationally connected variety X/k is strongly rationally connected ([KMM92b] 2.1,[Kol96] IV.3.9). Xu announced very recently ([Xu08]) that the smooth loci of log del Pezzo surfaces are not only rationally connected but also strongly rationally connected, which confirms a conjecture of Hassett and Tschinkel ([HT08], Conjecture 19).

In Chapter 1, we will give some preliminaries, including the definitions, and standard notation.

In Chapter 2, we first recall the definition of toric varieties and their properties in terms of lattices and fans. We will then introduce isogeny of toric varieties, which is the important trick in the proof of our main theorem.

In Chapter 3, we introduce rationally connected varieties as well as their arithmetic and geometric properties. We will also introduce rationally chain connected varieties, strongly rationally connected varieties, and the relations among the three classes. In the end of the chapter, we will discuss the role of free rational curves in the theory of rationally connected varieties. In this thesis, we define weakly free rational curves and geometrically free rational curves, which are generalizations of free rational curves.

In Chapter 4, we pose our main question (see §4.1). Then we get some results of free rational curves for projective spaces and quotient spaces. We have an affirmative answer for the problem for projective toric varieties over $\mathbb{C}$. As a corollary of the Main Theorem, we find that the smooth loci of toric varieties are strongly rationally connected. The rest of the chapter contains the proofs for the Main Lemma and Main Theorem.

It is expected that the main theorem can be solved by similar methods for complete toric varieties over any field.

1.2 Definitions and preliminaries

Throughout the paper, we are working over an uncountable field of characteristic 0, for simplicity $\mathbb{C}$. In this section, we recall some definitions and fix notation, which will be used throughout this thesis. Basic terminology can be found in [Ha77].

When we say that x is a point of a variety X, we mean that x is a closed point in X.

Definition 1.2.1. A variety X over a field k is called *rational* (or k-*rational* or *rational* over k) if X is k-birational to $\mathbb{P}_k^{\dim X}$. We say that X is *unirational* if there is a dominant map $\mathbb{P}_k^n \dashrightarrow X$.

Definition 1.2.2. A *rational curve* is a nonconstant morphism $f : \mathbb{P}^1 \to X$.

By the Lüroth theorem (See [Ha77] Chap IV. Example 2.5.5), every morphism $g : \mathbb{P}^1 \to X$ is either constant or it can be written as $\mathbb{P}^1 \xrightarrow{h} \mathbb{P}^1 \xrightarrow{f} X$ where f is birational onto its image, and h is a finite morphism.

$\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers, and similarly $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers.

A *morphism* of varieties is everywhere defined. It is denoted by a solid arrow $f : X \to Y$. A *rational map* of varieties is defined on a dense open set; it is denoted by a dotted arrow $f : X \dashrightarrow Y$.

For a birational morphism $f : Y \to X$, the *exceptional set* $\mathrm{Ex}(f) \subseteq Y$ is the set of points $\{y \in Y\}$ where f is not biregular (that is f^{-1} is not a morphism at $f(y)$).

A *contraction* of Y is a proper surjective morphism $f : Y \to X$ with $f_*\mathcal{O}_Y = \mathcal{O}_X$. We also say that it is an *extraction* of X, especially, if it is constructed from X. We say f is a *small contraction* if $\mathrm{Ex}(f)$ has codimension ≥ 2.

A topological space X is a *Zariski space* if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point. A subset of X is *locally closed* if it is the intersection of an open subset with a closed subset. A subset of X is *constructible* if it can be written as a finite disjoint union of locally closed subsets. See [Ha77] Chap II. Exercise 3.17, 3.18, 3.19.

Definition 1.2.3. Let X be a smooth variety, and $D = \cup_{i=1}^r D_i$ is a Weil divisor on X, where D_i are prime divisors. A point $P \in C \cap D$ is call a *n-point* if $P \in D_{i_1} \cap \cdots \cap D_{i_n}$ for some $1 \leq n \leq r$ and $P \notin D_{j_1} \cap \cdots \cap D_{j_{n+1}}$ for any $\{D_{j_1}, \ldots, D_{j_{n+1}}\} \subseteq \{D_1, \ldots, D_r\}$. If $n = 1$, we also say P is a *unary* point.

Let X be a smooth variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor (or $\mathbb{R}$-divisor, when $d_i \in \mathbb{R}$) on X. We say that D is a *simple normal crossing* divisor (abbreviated as SNC) if each D_i is smooth and they intersect everywhere transversally.

Let X be a scheme. A *resolution* of X is a proper birational morphism $g : Y \to X$ such that Y is smooth.

Definition 1.2.4. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety X, that is, $d_i \in \mathbb{Q}$ and D_i is a prime divisor on X for every i. Then D is said to be $\mathbb{Q}$-*Cartier* if there exists a positive integer m such that mD is a Cartier divisor. A normal variety X is said to be $\mathbb{Q}$-*factorial* if every normal prime divisor D on X is $\mathbb{Q}$-Cartier.

Let X be a scheme and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor (or $\mathbb{R}$-divisor) on X. A *log resolution* of (X, D) is a proper birational morphism $g : Y \to X$ such that Y is smooth, $\mathrm{Ex}(g)$ is a divisor and $\mathrm{Ex}(g) \cup g^{-1}(\mathrm{Supp}\ D)$ is an SNC divisor. Log resolutions exist for varieties over a field of characteristic zero by Theorem 1.2.5 (see [Hironaka64] or [KM98] Theorem 0.2).

Theorem 1.2.5. *Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed suchscheme Z. Then there are a smooth variety Y and a projective morphism $f : Y \to X$ such that*

(1) f is an isomorphism over $X \setminus (\mathrm{Sing}\ X \cup \mathrm{Supp}\ Z)$,

(2) $f^* I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and

(3) Ex $(f) \cup D$ is an SNC divisor.

Let (X, D) be a log pair with an $\mathbb{R}$-divisor D such that $K+D$ is $\mathbb{R}$-Cartier. Suppose that $f : Y \to X$ is a log resolution of (X, D). Then we have $K_Y + f_*^{-1} D = f^*(K+D) + \sum e_i E_i$ where E_i are f-exceptional prime divisors and we call $e_i := e(E_i, X, D)$ the *discrepancy* of (X, D) at E_i (the discrepancy of E_i can also be defined without a log resolution, but with a divisorial resolution of E_i only). The value $a(E_i, X, D) = e_i + 1$ is called the *log discrepancy* of (X, D) at E_i. If E is a prime non-exceptional divisor on X, then we define $a(E, X, D) = 1 - \mathrm{mult}_E D$. The pair (X, D) has only *log canonical* *(lc) singularities* if $e_i \geq -1$ for all exceptional and non-exceptional divisors E_i. We say that (X, D) is *Kawamata log terminal* (or *klt*), if all $e_i > -1$ and $\lfloor D \rfloor \leq 0$.

Definition 1.2.6. A smooth projective variety X is called *Fano* if the anti canonical divisor $-K_X$ is ample. A normal projective variety X is called $\mathbb{Q}$-*Fano* if $-K_X$ is $\mathbb{Q}$-Cartier and ample. If $\dim X = 2$, it is called a *del Pezzo surface*.

Definition 1.2.7. A normal projective variety X is called *FT* (*Fano Type*) if there exists an effective $\mathbb{Q}$-divisor D, such that (X, D) is klt (see §1.2 for definition) and $-(K_X + D)$ is ample. See [PSh09] Lemma-Definition 2.6. for other equivalent definitions.

Suppose T parameterizing a family of rational curves, and ev: $\mathbb{P}^1 \times T \to X$ is the evaluation morphism (map). Let $f_t = \mathrm{ev}|_{\mathbb{P}^1 \times t} : \mathbb{P}^1 \to X$. Let B be a set of closed points in $\mathbb{P}^1$, and $B_t = B \times t, t \in T$. If $B_t = B$ for all $t \in T$, we write $f_t|_B$ for $f_t|_{B_t}$.

Occasionally, we will apply the projection formula.

Theorem 1.2.8. *(projection formula, see [Debarre01] Chap I 1.9) Let $\pi : X \to Y$ be a proper morphism between varieties and let C be a curve on X. We define the 1-cycle $\pi_* C$ as follows: if C is contracted to a point by π, set $\pi_* C = 0$; if $\pi(C)$ is a curve on Y, set $\pi_* C = d \cdot \pi(C)$, where d is the degree of the morphism $C \to \pi(C)$ induced by π. If D is a Cartier divisor on X, we have*

$$\pi^* D.C = D.\pi_* C.$$

2 Toric varieties

2.1 Introduction

The theory of toric varieties was first introduced by Demazure (see [Demazure70]) and then by Mumford et al. (see [TE73]), Satake (see [Satake73]) and Miyake-Oda (see [MO73]). Now it is widely used in mathematics and theoretical physics. Our main references are from definitions in [Fulton93], [Oda78], [Oda88] and [Danilov78].

Definition 2.1.1. A *toric variety* X of dimension n is a normal variety that contains a torus $T = (\mathbb{C}^*)^n$ as a dense open subset, together with an action $T \times X \to X$ of T on X that extends the natural action of T on itself by multiplication.

Definition 2.1.2. A *toric morphism* or *equivariant morphism* consists of a homomorphism $f : T \to T'$ and a morphism $\bar{f} : X \to X'$ such that $\bar{f}(tx) = f(t)\bar{f}(x)$ for $t \in T, x \in X$.

The other way to define toric varieties is combinatorial using lattices and fans. Let $N \cong \mathbb{Z}^n$ be a lattice of rank n. Let $M = \mathrm{Hom}(N, \mathbb{Z})$ denote the dual lattice, with dual pairing denoted $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$. A *fan* Δ is a finite collection of convex cones $\sigma \subseteq N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying the following:

(i) Each convex cone $\sigma \in \Delta$ is rational polyhedral in the sense that there are finitely many $v_1, \ldots, v_s \in N \subseteq N_{\mathbb{R}}$ such that $\sigma = \{r_1 v_1 + \cdots + r_s v_s | r_i \in \mathbb{R}_{\geq 0} \text{ for all } i\}$ and is strongly convex in the sense that $\sigma \cap (-\sigma) = \{0\}$.

(ii) Each face τ of a convex cone $\sigma \in \Delta$ again belongs to Δ.

(iii) The intersection of two cones in Δ is a face of each.

If σ is a cone in N, the *dual cone* $\check{\sigma}$ is the set of vectors in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ that are nonnegative on σ. This determines a commutative semi-group

$$S_\sigma = \check{\sigma} \cap M = \{u \in M \mid \ \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}.$$

This semigroup is finitely generated (see [Fulton93] §1.2 Gordon's Lemma). Such an algebra corresponds to an affine variety: set

$$U_\sigma = \text{Spec }(\mathbb{C}[S_\sigma]) = \{u : S_\sigma \to \mathbb{C} \mid u(O) = 1, u(m+m') = u(m)u(m') \ \ \forall m, m' \in S_\sigma\}.$$

If τ is a face of σ (denoted by $\tau < \sigma$), then S_σ is contained in S_τ, so $\mathbb{C}[S_\sigma]$ is a subalgebra of $\mathbb{C}[S_\tau]$, which gives an open embedding $U_\tau \to U_\sigma$. With these identifications, these affine varieties can be glued from U_σ together to form an algebraic variety, which is called the *toric variety* associated to the fan Δ, denoted by $X(\Delta)$. The big torus of $X(\Delta)$ is

$$T_N = U_0 = \text{Spec }(\mathbb{C}[M]) = \text{Spec }\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = \mathbb{C}^* \times \cdots \times \mathbb{C}^*.$$

A map of fans $\varphi : (N', \Delta') \to (N, \Delta)$ is a $\mathbb{Z}$-linear homomorphism $\varphi : N' \to N$ whose scalar extension $\varphi_\mathbb{R} : N'_\mathbb{R} \to N_\mathbb{R}$, where $N'_\mathbb{R} = N' \otimes_\mathbb{Z} \mathbb{R}, N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$, satisfies the following property: For each $\sigma' \in \Delta'$ there exists $\sigma \in \Delta$ such that $\varphi_\mathbb{R}\sigma' \subset \sigma$.

Theorem 2.1.3. *([Oda88] §1.5, Theorem 1.13) A map of fans $\varphi : (N', \Delta') \to (N, \Delta)$ gives rise to a morphism of algebraic varieties*

$$\varphi_* : X = X(\Delta') \to Y = Y(\Delta),$$

whose restriction to the open subset $T_{N'}$ coincides with the homomorphism of algebraic tori

$$\varphi \otimes 1 : T_{N'} = N' \otimes_\mathbb{Z} \mathbb{C}^\times \to T_N = N \otimes_\mathbb{Z} \mathbb{C}^\times$$

arising from φ. Through this homomorphism, φ_* is equivariant with respect to the actions of $T_{N'}$ and T_N on the toric varieties.

Conversely, suppose $f' : T_{N'} \to T_N$ is a homomorphism of algebraic tori and $f : X = X(\Delta') \to Y = Y(\Delta)$ is a morphism equivariant with respect to f'. Then *there exists a unique $\mathbb{Z}$-linear homomorphism $\varphi : N' \to N$ which gives rise to a map of fans $(N', \Delta') \to (N, \Delta)$ such that $f = \varphi_*$.*

The first way of defining toric varieties is from a geometric point of view. The second one is from a combinatorial point of view. The previous lemma essentially establishes that they are equivalent. See also [Oda78] and [Fulton93].

The gist of the theory lies in the fact that geometric properties of toric varieties can very often be described in terms of the elementary geometry of fans. Here we give some criteria that we will use later.

Theorem 2.1.4. *([Oda88] §1.4 Theorem 1.10)(Nonsingularity) The toric variety $X(\Delta)$ is nonsingular if and only if each $\sigma \in \Delta$ is nonsingular in the following sense: There exist a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N and $s \leq r$ such that $\sigma = \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_s$.*

Theorem 2.1.5. *([Oda88] §1.4 Theorem 1.11)(Completeness) The toric variety $X(\Delta)$ is complete if and only if Δ is a finite and complete fan, i.e., Δ is a finite set with the support $|\Delta| := \cup_{\sigma \in \Delta} \sigma$ coinciding with the entire $N_{\mathbb{R}}$.*

Theorem 2.1.6. *([Oda88] §1.5 Theorem 1.15)(Proper morphism) Let $\varphi : (N', \Delta) \to (N, \Delta)$ be a map of fans. The equivariant morphism $f = \varphi_* : X(\Delta') \to Y(\Delta)$ is proper (i.e., the inverse image of each compact subset is compact) if and only if for each $\sigma \in \Delta$, the set $\Delta'_\sigma = \{\sigma' \in \Delta \mid \varphi(\sigma') \subseteq \sigma\}$ is finite and $\varphi^{-1}(\sigma) = |\Delta'_\sigma| := \cup_{\sigma' \in \Delta'_\sigma} \sigma'$.*

Theorem 2.1.7. *([Oda88] §1.5 Corollary 1.17) The equivariant morphism $\varphi_* : X(\Delta') \to Y(\Delta)$ is proper if and only if $\varphi : N' \to N$ is an isomorphism and Δ' is a locally finite subdivision of Δ under the identification $N'_{\mathbb{R}} = N_{\mathbb{R}}$.*

As with any set on which a group acts, a toric variety $X = X(\Delta)$ is a disjoint union of its orbits by the action of the torus $T = T_N$. We can describe the T_N-orbits in terms of Δ as follows:

Theorem 2.1.8. *([Oda88] §1.2 Proposition 1.6) For each $\sigma \in \Delta$ we can regard the quotient algebraic torus*

$$orb(\sigma) := \{u : M \cap \sigma^{\perp} \to \mathbb{C}^*;\ \text{group homomorphisms}\}$$

of T_N as a T_N-orbit in $X(\Delta)$. Every T_N-orbits is of this form and in this way, Δ is in one-to-one correspondence with the set of T_N-orbits in $X(\Delta)$. Moreover, the following hold:

(i) $orb(\{O\}) = U_O = T_N$.

(ii) For $\sigma \in \Delta$, the complex dimension of $orb(\sigma)$ coincides with the codimension $r - \dim \sigma$ of σ in $N_{\mathbb{R}}$.

(iii) For $\sigma, \tau \in \Delta$, τ is a face of σ if and only if $orb(\sigma)$ is contained in the closure of $orb(\tau)$.

(iv) For $\sigma \in \Delta$, $orb(\sigma)$ is the unique closed T_N-orbit in U_σ and we have $U_\sigma = \coprod_{\tau < \sigma} orb(\tau)$.

We also need two facts about $\mathbb{Q}$-factoriality of toric varieties.

Theorem 2.1.9. *([Matsuki02] §14.1 Lemma 14-1-1) A toric variety $X(\Delta)$ is $\mathbb{Q}$-factorial if and only if each $\sigma \in \Delta$ is simplicial, i.e., there exist exactly $s = \dim \sigma$ lattice points $v_1, \ldots, v_s$ such that $\sigma = \{r_1 v_1 + \cdots + r_s v_s | r_i \geq 0\}$. It is factorial if and only if it is nonsingular, i.e., $v_1, \ldots, v_s$ form a (part of a) $\mathbb{Z}$-basis of N.*

Theorem 2.1.10. *([Fj03] Corollary 5.9. Small projective toric $\mathbb{Q}$-factorialization). Let X be a toric variety over k. Then there exists a small projective toric morphism $f : \tilde{X} \to X$ such that $\tilde{X}$ is $\mathbb{Q}$-factorial over k.*

2.2 Isogeny of toric varieties

The following lemma is a characterization of finite surjective toric morphisms with respect to lattices and fans of toric varieties.

Lemma 2.2.1. *Let $f : X \to Y$ be a toric morphism, and $\varphi : (N', \Delta') \to (N, \Delta)$, where (N', Δ') (respectively (N, Δ)) is the lattice and fan of X (respectively of Y), be its corresponding map of lattices and fans. Then f is finite surjective if and only if*

1) rank N'= rank $\varphi(N')$ = rank N.

2) $\varphi_{\mathbb{R}}$ gives a one-to-one correspondence between $\sigma' \in \Delta'$ and $\sigma \in \Delta$, and we denote this by $\varphi_{\mathbb{R}}\Delta' = \Delta$.

Remark. Let $f : X \to Y$ be a finite surjective toric morphism of toric varieties and $\varphi : (N', \Delta') \to (N, \Delta)$ be the corresponding map of fans, then we can identify N' as a sublattice of N and $\Delta' = \Delta$.

Proof. **If** part:

Since rank N' = rank $\varphi(N')$ and $N'/\ker\varphi \cong \varphi(N')$, we have $\ker\varphi_{\mathbb{R}} = 0$. So, φ induces a toric isomorphism between (N', Δ') and $(\varphi(N'), \varphi_{\mathbb{R}}\Delta)$. On the other hand, φ gives a one-to-one correspondence between $\sigma' \in \Delta'$ and $\sigma \in \Delta$, i.e., $\varphi_{\mathbb{R}}\Delta' = \Delta$. Therefore, after an identification, X can be viewed as a toric variety given by (N', Δ), where $N' \subseteq N$ is an inclusion of same rank lattices. Therefore we get that f is a finite surjective toric morphism [cf [Fulton93] page 33-34]. More precisely, the finite group N/N' acts on X, and $X \to Y$ is the quotient map.

Only if part:

1) Since f is a finite surjective toric morphism, we get $\dim X = \dim f(X) = \dim Y$. Therefore rank N' =rank $\varphi(N')$ =rank N.

2) The lattice map φ can be decomposed as:

$$\begin{array}{ccc} (N', \Delta') & \xrightarrow{\quad\quad\varphi\quad\quad} & (N, \Delta) \\ & {\scriptstyle\varphi}\searrow \qquad \nearrow{\scriptstyle \mathrm{Id}} & \\ & (N, \varphi_{\mathbb{R}}\Delta') & \end{array}$$

where Id is the identity map of the lattice N. This gives a decomposition of f into toric morphisms:

$$X \xrightarrow{g} Z \xrightarrow{h} Y \ ,$$

where Z is the toric variety given by $(N, \varphi_{\mathbb{R}}\Delta')$.

Since f is finite, it is proper. By the criterion of properness of toric morphisms (See Theorem 2.1.6), $\varphi_{\mathbb{R}}\Delta'$ is a subdivision of Δ. Therefore, by the criterion of birational maps between toric varieties (See Theorem 2.1.7), the morphism h is birational. On the other hand, since f is finite and surjective, we get h is finite and surjective as well. Therefore, the toric morphism h has to be an isomorphism since h is finite surjective and birational. Hence $\varphi_{\mathbb{R}}\Delta' = \Delta$. □

Lemma 2.2.2. *Let N' be a sublattice of N and rank N' = rank N, then there exists a positive integer r, such that rN is a sublattice of N'*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of the lattice N and $\{f_1, \ldots, f_n\}$ a basis of N', where n =rank N =rank N'. Let $F = (f_1, \ldots, f_n)^\top, E = (e_1, \ldots, e_n)^\top$, where $^\top$ denotes the transpose of matrix.

Since N' is a sublattice of N, and rank N' = rank N, we get $F = AE$ for an invertible matrix A over integers, i.e., $A \in GL_n(\mathbb{Z})$. So we have $(\det A)E = A^*F$, where A^* is the adjoint matrix of A. Let $r = |\det A| \neq 0$ since A is invertible, which is a positive integer. Therefore, rN is a sublattice of N'. □

Lemma 2.2.3. *For a lattice N, fan Δ and a positive integer r, let $\varphi : (rN, \Delta) \to (N, \Delta)$ be a map of fans which is defined by $\varphi(a) = a/r$ for $a \in rN$. Then φ induces* *an isomorphic toric morphism $f : X \to Y$ where X is the toric variety given by (rN, Δ) and Y is the toric variety given by (N, Δ).*

Proof. Let $\psi : (N, \Delta) \to (rN, \Delta)$ be a map of lattices and fans which is defined by $\psi(a) = ar$ for $a \in N$. Then we can easily verify that $\psi\varphi = \mathrm{Id}_{rN}$ and $\varphi\psi = \mathrm{Id}_N$. Therefore, the corresponding toric morphisms $f : X \to Y$ and $g : Y \to X$ satisfy $fg = \mathrm{Id}_Y$ and $gf = \mathrm{Id}_X$. In other words, the toric morphism f, which is induced by φ, is a toric isomorphism. □

Corollary 2.2.4. *Let $f : X \to Y$ be a finite surjective toric morphism. Then there exists a finite surjective toric morphism $g : Y \to X$.*

Proof. By the remark of Lemma 2.2.1, we can identify the lattice N' of X as a sublattice of the lattice N of Y. And identify the fan Δ' of X as the fan Δ of Y.

By Lemma 2.2.2, there exists a positive integer r, such that rN is a sublattice of N'. Therefore the inclusion map of lattices $\phi : (rN, \Delta) \to (N, \Delta)$ induces a finite surjective toric morphism $g : Z \to X$, where Z is the toric variety given by (rN, Δ).

By Lemma 2.2.3, Z and Y are isomorphic as toric varieties. Then, after identification, $g : Y \to X$ is a finite surjective toric morphism. □

Definition 2.2.5. An *isogeny* of toric varieties is a finite surjective toric morphism. Toric varieties X and Y are said to be *isogenous* if there exists an isogeny $X \to Y$. The *isogeny class* of a toric variety X is a set consisting of all toric varieties Y such that X and Y are isogenous.

Let us sum up the properties of isogeny.

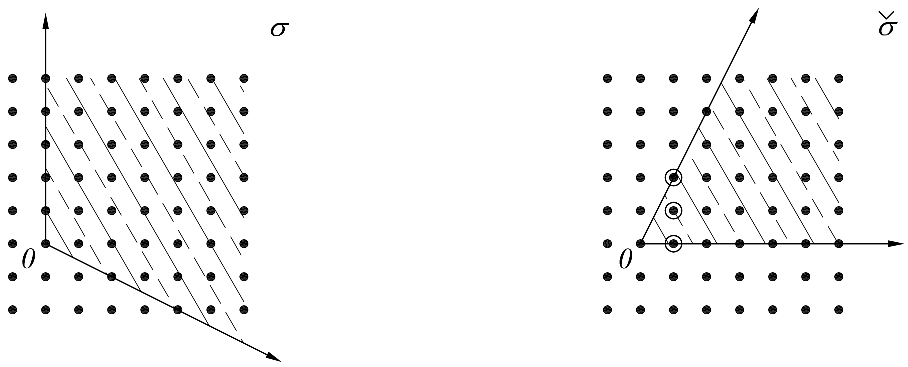
1) If toric varieties X and Y are isogenous, then Y and X are isogenous by Corollary 2.2.4. Moreover, isogeny is an equivalence relation, i.e., it is reflexive, symmetric and transitive.

2) If a toric variety Y is in the isogeny class of X and $\mu : X \to Y$ is the isogeny, then there is a one-to-one correspondence between the set of orbits $\{O_i^X\}$ of X and the set of orbits $\{O_i^Y = \mu(O_i^X)\}$ of Y. Hence $\dim O_i^X = \dim O_i^Y$ for all i, and the number of orbits is independent of the choice of toric varieties in the isogeny class of X.

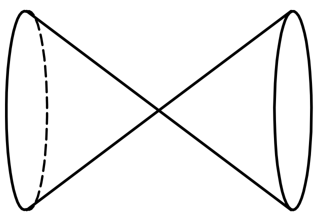
2.3 Examples

Example 2.3.1. Dimension 1 toric varieties are either $\mathbb{C}$, $\mathbb{C}^*$, or $\mathbb{CP}^1$.

Example 2.3.2. (singular toric surface) Take $n = 2$, and take σ generated by e_2 and $2e_1 - e_2$.



Semigroup generators for S_σ are $e_1^*, e_1^*+e_2^*$ and $e_1^*+2e_2^*$, so $\mathbb{C}[S_\sigma] = \mathbb{C}[X, XY, XY^2] = \mathbb{C}[U, V, W]/(V^2 - UW)$. Hence U_σ is a quadric cone, i.e., a cone over a conic.



Example 2.3.3. (Twisted or weighted projective space) Let vectors $v_1, \ldots, v_n$ be the standard basis $e_1, \ldots, e_n$ for $N = \mathbb{Z}^n$, and $v_0 = -e_1 - \cdots - e_n$. Let Δ be the fan

whose cones are generated by any proper subsets of the vectors $v_0, \ldots, v_n$. The toric variety $X(\Delta)$ is $\mathbb{CP}^n$.

To construct weighted projective space $\mathbb{P}(d_0, \ldots, d_n)$, where $d_0, \ldots, d_n$ are any positive integers, start with the same fan used in the construction of projective space, i.e., its cones generated by proper subsets of $\{v_0, v_1, \ldots, v_n\}$. However, the lattice N is taken to be generated by the vectors $(1/d_i)v_i, 0 \leq i \leq n$. The resulting toric variety is in fact the variety

$$\mathbb{P}(d_0, \ldots, d_n) = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts by $\zeta \cdot (x_0, \ldots, x_n) = (\zeta^{d_0} x_0, \ldots, \zeta^{d_n} x_n)$.

Example 2.3.4. Any toric variety X is rational, because it is birational to its big torus $(\mathbb{C}^*)^n$, which is rational, where $n = \dim X$.

Example 2.3.5. Projective toric varieties are FT (see Definition 1.2.7). Let K be the canonical divisor of the projective toric variety $X(\Delta)$, T be the torus of X, and $\Sigma = X \setminus T = \sum D_i$ be the complement of T in X. Then K is linearly equivalent to $-\Sigma$. Since X is projective, there is an ample invariant divisor L. Suppose that $L = \sum d_i D_i$. Let the polytope $\Box_L = \{m \in M | \langle m, e_i \rangle + d_i \geq 0, \forall e_i \in \Delta(1)\}$, where M is the dual lattice of N, and $\Delta(1)$ is the set consisting of 1-dimensional cones in Δ. Let u be an element in the interior of $\Box_L$. Then $D =$div $\chi^u + L$ is effective and ample and has support Σ, that is, $D = \sum d_i' D_i$ and all $d_i' > 0$.

Let ϵ be a positive rational number, such that all coefficients of prime divisors in ϵD are strictly less than 1. Then $\Sigma - \epsilon D$ is effective. It is easy to check that $(X, \Sigma - \epsilon D)$ is klt (see §1.2 for terminology and notation), and $-(K + \Sigma - \epsilon D) \sim \epsilon D$ is ample. Hence X is FT.

3 Rationally connected varieties

3.1 Rationally connected varieties

Definition 3.1.1. ([Kol96] Chap IV, 3.2 Definition.) Let X be a variety (not necessarily proper).

(1) We say that X is *rationally chain connected* if there is a family of proper and connected algebraic curves $g : U \to Y$ whose geometric fibers have only rational components with cycle morphism $u : U \to X$ such that

$$u^{(2)} : U \times_Y U \to X \times X \text{ is dominant.}$$

(The image of $u^{(2)}$ consists of those pairs $x_1, x_2 \in X$ such that $x_1, x_2 \in u(U_y)$ for some $y \in Y$.)

(2) We say that X is *rationally connected* if there is a family of proper algebraic curves $g : U \to Y$ whose geometric fibers are irreducible rational curves with cycle morphism $u : U \to X$ such that $u^{(2)}$ is dominant.

Roughly speaking, a variety X is rationally connected if for any two generic points $x_1, x_2 \in X$, there is a rational curve connecting x_1 and x_2, and the curve belongs to a bounded family. A variety X is rationally chain connected if for any two generic points $x_1, x_2 \in X$, there is a chain of rational curves connecting x_1 and x_2.

Remark. The affine space $\mathbb{A}^n$ is rational, but not rationally connected.

Motivation: Rationally connected varieties are right higher dimensional analogs of rational curves and rational surfaces. In other words, a rationally connected variety is expected to have plenty of rational curves. (see [Kol01].)

Theorem 3.1.2. *(see [AK03] Definition-Theorem 29) Let X be a smooth projective variety over $\mathbb{C}$. Then the following are equivalent:*

(1) There is a dense open set $X^0 \subseteq X$ such that for every $x_1, x_2 \in X^0$, there is a chain of rational curves connecting x_1 and x_2.

(1') There is a dense open set $X^0 \subseteq X$ such that for every $x_1, x_2 \in X^0$, there is a rational curve connecting x_1 and x_2.

(2) For every $x_1, x_2 \in X$ there is a rational curve through x_1 and x_2.

(3) For every $x_1, \ldots, x_m \in X$ there is a rational curve through $x_1, \ldots, x_m$.

(4) For every $x_1, \ldots, x_m \in X$ there is a free rational curve (see definition 3.2.2) through $x_1, \ldots, x_m$.

(5) There is a very free rational (see definition 3.2.2) curve on X.

Corollary 3.1.3. *([Kol96] Chap IV. 3.10.3 Theorem.) Let X be a smooth variety over $\mathbb{C}$. Then X is rationally chain connected if and only if X is rationally connected.*

Complete rationally connected varieties have many nice geometric and arithmetic properties (see [AK03] for an elementary introduction):

0) (Birational invariance) Let X and Y be two complete varieties with X birational to Y. Then X is rationally connected if and only if Y is rationally connected.

1) (Castelnuovo's criterion) A smooth projective rationally connected variety X satisfies $H^0(X, (\Omega^1_X)^{\otimes m}) = 0$ for every $m \geq 1$, and the converse holds for $\dim X = 1$, $\dim X = 2$ (Castelnuovo's criterion), and $\dim X = 3$ (see [KMM92b]). The converse is conjectured to hold for $\dim X \geq 4$.

2) (Deformation Invariance) Let S be a connected $\mathbb{C}$-scheme and $p : X \to S$ a smooth projective morphism. If a fiber X_s is rationally connected for some $s \in S$, then every fiber of p is rationally connected (see [KMM92b]). It is an open problem. Deformation invariance is expected to fail for the class of smooth non-unirational varieties or smooth non-rational varieties for $\dim X \geq 4$.

3) If X is a smooth variety, and there is a dominant morphism $X \to Y$ to a

rationally connected variety Y such that general fibers are rationally connected, then X itself is rationally connected, see [GrHaSt03].

4) (Algebraically simply connected, see [Debarre01] Chap 4, Corollary 4.18) Let X be a smooth projective rationally connected variety. Then any connected finite étale cover of X is trivial. The variety X is said to be *algebraically simply connected*.

5) (Simply connected, see [Debarre01] Chap 4, Corollary 4.18) Let X be a smooth projective rationally connected variety over $\mathbb{C}$, then the variety X is simply connected.

Example 3.1.4. It is easy to see that rationally connected varieties are rationally chain connected varieties. On the other hand, we have seen that for smooth projective varieties over $\mathbb{C}$, rationally chain connected varieties are also rationally connected. In general, the property of rational chain connectedness does not imply rational connectedness. For example, a projective cone X over a smooth elliptic curve E is rationally chain connected but X is not rationally connected. Actually, for any projective cone X over a smooth curve C with genus $g(C) \geq 1$, X is rationally chain connected but not rationally connected. However, if X has mild singularities (e.g. klt), then rational chain connectedness implies rational connectedness, see [Sh00].

Here are some examples of rationally connected varieties.

Example 3.1.5. Let $\mathfrak{A}$ be the set of projective rational varieties; $\mathfrak{B}$ the set of projective unirational varieties, and $\mathfrak{C}$ the set of rationally connected varieties. Then $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{C}$.

For the surfaces case, there is no difference, i.e., a projective rationally connected surface is rational, and thus unirational.

There are examples of threefolds X which are unirational but not rational. Whether or not $\mathfrak{B} = \mathfrak{C}$ is an open problem but they are expected to be unequal.

Example 3.1.6. All complete toric varieties are rational (see Example 2.3.4). Hence complete toric varieties are all rationally connected.

Example 3.1.7. Smooth Fano (see Definition 1.2.6) varieties are rationally connected. See [KMM92a].

Example 3.1.8. A smooth hypersurface X in $\mathbb{P}^n$ is rationally connected if and only if $\dim X \leq n$.

Example 3.1.9. The closure of the image of a rationally connected variety is rationally connected.

Example 3.1.10. FT varieties are rationally connected. See [Sh00],[HM07],[Zh06].

Example 3.1.11. Any log canonical Fano variety is rationally chain connected [Sh00].

3.2 Free rational curves and strongly rationally connected varieties

Definition 3.2.1. Let C be a proper curve without embedded points, X a smooth variety and $f : C \to X$ a morphism. Let $B \subseteq C$ be a closed subscheme with ideal sheaf I_B and $g = f|_B$.

f is called *free over* g if f is nonconstant and the following two equivalent conditions are satisfied:

(1) $H^1(C, f^*T_X \otimes J) = 0$ for every subsheaf $J \subseteq I_B$ such that $\text{length}(I_B/J) = 1$. (If C is smooth, then this can be written as: $H^1(C, f^*T_X \otimes I_B(-p)) = 0$ for every $p \in C$.)

(2) $H^1(C, f^*T_X \otimes I_B) = 0$ and $f^*T_X \otimes I_B$ is generated by global sections.

Definition 3.2.2. In the $C = \mathbb{P}^1$ case, by [Ha77] V. Exercise 2.6, we can write $f^*T_X \otimes I_B \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$. $f : \mathbb{P}^1 \to X$ is called a *free rational curve* (resp. *very free rational curve*) *over* g if all $a_i \geq 0$ (resp. all $a_i \geq 1$.)

Definition 3.2.3. ([Kol96] Chap II 3.8 Definition) Let E be a vector bundle on $\mathbb{P}^1$. E is called semi-positive (resp. ample) if any of the following equivalent conditions is satisfied:

(1) $H^1(\mathbb{P}^1, E(-1)) = 0$ (resp. $H^1(\mathbb{P}^1, E(-2)) = 0$);

(2) E (resp. $E(-1)$) is generated by global sections;

(3) $E \cong \sum \mathcal{O}(a_i)$ and $a_i \geq 0$ (resp. ≥ 1) for every i;

(4) $H^0(\mathbb{P}^1, E) \to E \otimes k(p)$ (resp. $H^0(\mathbb{P}^1, E(-1)) \to E(-1) \otimes k(p)$) is surjective for some $p \in \mathbb{P}^1$.

Remark. Intuitively, a free curve can be moved in every direction, while a very free curve moves freely in any direction with one point fixed.

Example 3.2.4. ([Debarre01] Chap 4, Examples 4.7) (1) If $f : \mathbb{P}^1 \to X$ is nonconstant, f^*T_X contains $T_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$ and $a_1 \geq 2$. So if $f : \mathbb{P}^1 \to X$ is a free rational curve, $K_X \cdot f_*\mathbb{P}^1 = -\sum_{i=1}^n a_i \leq -2$. There are no free rational curves on a variety whose canonical divisor is nef.

(2) Any rational curve in $\mathbb{P}^n$ is very free.

(3) A smooth rational curve C on a smooth surface X is free (resp. very free) if and only if $C^2 \geq 0$ (resp. $C^2 > 0$).

The following proposition shows that a free curve can move from any smooth subvariety of codimension at least 2.

Proposition 3.2.5. *([Kol96] Chap II. 3.7 Proposition.) Let $f : C \to X$ be free over*

g. Let $Z \subset X$ be a subscheme of codimension at least two. Then

$$f'(C - B) \cap Z = \emptyset$$

for a general deformation f' of f over g.

The following theorem is very useful for determining if a rational curve is free.

Theorem 3.2.6. *([Kol96] Chap II. 3.11 Theorem) Let X be a smooth projective variety over a field of characteristic 0. Let $B \subset \mathbb{P}^1$ be a subscheme, $|B| \leq 2$ and $g : B \to X$ a morphism. There are countably many subvarieties $V_i = V_i(B, g) \subsetneq X$ such that if $f : \mathbb{P}^1 \to X$ is a nonconstant morphism such that $f|_B = g$ and* im *$f \not\subset \cup V_i$, then f is free over B.*

We also need to introduce the deformations of a tree of rational curves, which is a main trick Kollar-Miyaoka-Mori used in their foundational work on rationally connected varieties (see [KMM92a]).

Definition 3.2.7. ([Kol96] Chap II. 7.1 Definition) Let $f : C \to X$ be a morphism. A *smoothing* of $f : C \to X$ is a diagram

$$\begin{array}{ccc} Y & \xrightarrow{F} & X \\ {\scriptstyle q}\downarrow & & \\ 0 \in T & & \end{array}$$

where $0 \in T$ is a connected, smooth and pointed curve such that

(1) q is flat and proper over T;

(2) q is smooth over $T - \{0\}$;

(3) $[F|_{q^{-1}(0)} : q^{-1}(0) \to X] \cong [f : C \to X]$.

$F|_{q^{-1}(t)} : q^{-1}(t) \to X$ is called a *nearby smoothing* for $t \neq 0$.

Definition 3.2.8. ([Kol96] Chap II. 7.2 Definition) Let $f : C \to X$ be a morphism and $x_1, \ldots, x_n \in C$ distinct points. We say that the above smoothing fixes $f(x_1), \ldots, f(x_n)$ if there are sections $t_i : T \to Y$ such that

(1) $t_i(0) = x_i \in C \subset Y$, and

(2) $F \circ t_i(T) = f(x_i)$ for every i.

We say that f is *smoothable fixing* $f(x_1), \ldots, f(x_n)$ if there is a smoothing which fixes $f(x_1), \ldots, f(x_n)$.

Definition 3.2.9. ([Kol96] Chap II. 7.4 Definition) Let C be a connected curve and $D \subset C$ a subcurve, C_i the irreducible components of C not contained in D. We say that C is *obtained from D by attaching trees* if

(1) every C_i is isomorphic to $\mathbb{P}^1$, and

(2) C_i intersects $D + \sum_{j=1}^{i-1} C_j$ in a single point which is an ordinary node of C.

(3) If $D \cong \mathbb{P}^1$, then we say that C is a *tree of smooth rational curves.*

(4) A tree of smooth rational curves is called a *chain of smooth rational curves* (of length n) if C_{i+1} intersects $\sum_{j=1}^{i} C_j$ in a single point which is a point of $C_i (1 \leq i \leq n-1)$.

Theorem 3.2.10. *([Kol96] Chap II. 7.6 Theorem) Let $C = \sum C_i$ be a tree of smooth rational curves, and $f : C \to X$ a morphism such that X is smooth along $f(C)$.*

(1) Fix a smooth point $0 \in C$. If $f^*T_X|_{C_i}$ is semi-positive for every C_i, then $f : C \to X$ can be smoothed keeping $f(0)$ fixed.

(2) Fix smooth points $p_i \in C_i$. If $f^*T_X|_{C_i}$ is ample for every C_i, then $f : C \to X$ can be smoothed keeping the $f(p_i)$ fixed.

(3) Let $f_t : \mathbb{P}^1 \to X$ be a nearby smoothing. Then $f_t^*T_X$ is semi-positive in the first case and ample in the second case.

Definition 3.2.11. ([Kol96] Chap II 7.7 Definition) A *comb with n teeth* is a connected and reduced curve C with irreducible components $D, C_1, \ldots, C_n$ with the following properties:

(1) D is smooth and every C_i is isomorphic to $\mathbb{P}^1$; (D is called the handle and C_i the teeth)

(2) the only singularities of C are ordinary nodes;

(3) every C_i intersects D in a single point x_i and $x_i \neq x_j$ for $i \neq j$. These are the only intersections between the irreducible components.

A *subcomb* of C is a subcurve which contains the handle.

Theorem 3.2.12. *([Kol96] Chap II 7.9 Theorem) Let $C = D + \sum_{1 \leq i \leq m} C_i$ be a comb, $p_1, \ldots, p_k \in D$ points different from the x_i. Let $f : C \to X$ be a morphism such that X is smooth along $f(C)$ and $f^*T_X|_{C_i}$ is semi-positive for every i. Then there is a subcomb $C' = D + \sum' C_i$ with m' teeth such that $f' := f|_{C'} : C' \to X$ is smoothable fixing $f'(p_1), \ldots, f'(p_k)$ and m' satisfies both of the following inequalities:*

$$m' \geq m - h^1(D, (f|_D)^*T_X(-\sum p_i)), \text{ and}$$
$$m' \geq m - (K_X.D) - k \dim X + \dim X \cdot \chi(\mathcal{O}_D) - \dim_{[f|_D]} Hom(D, X, f|_{p_1,\ldots,p_k}).$$

The theorem can be applied to deform a nonfree rational curve to be a free rational curve, if $D = \mathbb{P}^1$. If we attach to the curve $D = \mathbb{P}^1$ sufficiently many free rational curves (e.g. $m \geq (K_X.D) + k \dim X = \dim X \cdot \chi(\mathcal{O}_D) + \dim_{[f|_D]} \mathrm{Hom}(D, f|_{p_1,\ldots,p_k})$ and $m \geq h^1(D, (f|D)^*T_X(-\sum p_i))$, then $D + \sum' C_i$ can be smoothed to a rational curve. Then apply Theorem 3.2.6; it is free.

Now let's introduce the definition of a strongly rationally connected variety.

Theorem 3.2.13. *([Kol96] Chap IV, 3.9.4Theorem.) Let X be a smooth separably rationally connected variety over an algebraically closed field. There is a unique largest* *open subset $\emptyset \neq X^0 \subset X$ such that if $x_1, \ldots, x_m \in X^0$ are distinct closed points, then there is a morphism $f : \mathbb{P}^1 \to X^0$ such that*

(1) $x_1, \ldots, x_m \in f(\mathbb{P}^1)$;

(2) f^*T_X is ample;

(3) f is an immersion if $\dim X = 2$ and an embedding if $\dim X \geq 3$.

Moreover, if $C \subset X$ is a rational curve such that $C \cap X^0 \neq \emptyset$, then $C \subset X^0$.

With the same notation as above, X is called strongly rationally connected if $X^0 = X$. This new concept was first introduced by B. Hassett and Y. Tschinkel. There are no examples where $X^0 \neq X$.

Definition 3.2.14. (see [HT08] Definition 14.) A smooth separably rationally connected variety Y is *strongly rationally connected* if any of the following conditions hold:

(1) for each point $y \in Y$, there exists a rational curve $f : \mathbb{P}^1 \to Y$ joining y and a generic point in Y;

(2) for each point $y \in Y$, there exists a free rational curve containing y;

(3) for any finite collection of points $y_1, \ldots, y_m \in Y$, there exists a very free rational curve containing the y_j as smooth points;

(4) for any finite collection of jets

$$\text{Spec } k[\epsilon]/\langle \epsilon^{N+1} \rangle \subset Y, \quad i = 1, \ldots, m$$

supported at distinct points $y_1, \ldots, y_m$, there exists a very free rational curve smooth at $y_1, \ldots, y_m$ and containing the prescribed jets.

Example 3.2.15. The example follows from Zhang ([Zh95]) that the smooth locus of a rationally connected variety may not be strongly rationally connected.

Let $X = (\mathbb{P}^1 \times E)/\langle g \rangle$ where E is a smooth elliptic curve and g is an involution acting diagonally and effectively on both factors (with $g|_E$ = involution). Then X has 8 singular points. The second projection gives a fibration $X \to E/\langle g \rangle = \mathbb{P}^1$ (I use the same g to denote the action on E) with general fibres $\mathbb{P}^1$, so X is a rational surface. The smooth locus X^0 of X has an étale double cover equal to $\mathbb{P}^1 \times E$ minus a few points, so $\pi_1(X^0)$ is infinite. However, if Y is a smooth rationally connected variety (possibly not be proper), then $\pi_1(Y)$ is finite (see Lemma 7.8 [KM99] and Proposition 2.10 [Kol95] for example).

This example is not FT. Otherwise its smooth locus X^0 is rationally connected (see [Xu08]).

3.3 Weakly free and geometrically free rational curves

Definition 3.3.1. Let X be a complete normal variety, B be a set of finitely many closed points in $\mathbb{P}^1$, and $g : B \to X$ be a morphism. A rational curve $f : \mathbb{P}^1 \to X$ is called *weakly free* over g if there exist an irreducible family of rational curves T and an evaluation morphism ev: $\mathbb{P}^1 \times T \to X$ such that

1) $f = f_{t_0} = \mathrm{ev}|_{\mathbb{P}^1 \times t_0}$ for some $t_0 \in T$,

2) for any $t \in T$, $f_t = \mathrm{ev}|_{\mathbb{P}^1 \times t}$ is a rational curve and $f_t|_B = g$,

3) the evaluation morphism ev: $\mathbb{P}^1 \times T \to X$ by $\mathrm{ev}(x, t) = f_t(x)$ is dominant.

We say that $f' : \mathbb{P}^1 \to X$ is a *general deformation* of f, or f' is a *sufficiently general weakly free rational curve*, if there is an open subset U of T, such that $f' = f_t$ and $t \in U \subseteq T$. We say that $g : \mathbb{P}^1 \to X$ is a *general deformation of* f, *which is a weakly free rational curve*, if there is an irreducible family T', $g = g_t$ for some $t \in T'$ and g is weakly free with its own family.

Definition 3.3.2. Let X be a complete normal variety, B be a set of finitely many closed points in $\mathbb{P}^1$, and $g : B \to X$ be a morphism. A rational curve $f : \mathbb{P}^1 \to X$ is called *geometrically free* over g if there exist an irreducible family of rational curves T and an evaluation morphism ev: $\mathbb{P}^1 \times T \to X$ such that

1) $f = f_{t_0} = \mathrm{ev}|_{\mathbb{P}^1 \times t_0}$ for some $t_0 \in T$,

2) for any $t \in T$, $f_t = \mathrm{ev}|_{\mathbb{P}^1 \times t}$ is a rational curve and $f_t|_B = g$,

3) for any codimension 2 subvariety Z in X, $f_t(\mathbb{P}^1) \cap Z \subseteq g(B)$ for general $t \in T$ (general meaning t belongs to a dense open subset in T, depending on Z).

If X is smooth over an uncountable field of characteristic 0, then weak freeness over g is equivalent to usual freeness over g if $|B| \leq 2$.

Proposition 3.3.3. *For a normal algebraic variety X, geometrical freeness implies weak freeness.*

Proof. Let T be the irreducible family of the geometrically free rational curve $f : \mathbb{P}^1 \to X$. Let $Y = \overline{\mathrm{ev}(\mathbb{P}^1 \times T)}$ be the closure of $\mathrm{ev}(\mathbb{P}^1 \times T)$, where ev: $\mathbb{P}^1 \times T \to X$ is the evaluation morphism. So Y is irreducible and has codimension ≤ 1.

Suppose that Y is of codimension 1. Let D be a general big divisor in Y, such that $f'(\mathbb{P}^1) \cap D \neq \emptyset$ for a general deformation f' of f. Indeed, by Chow's lemma, there is a projective variety Y' and a morphism $g : Y' \to Y$, such that g induces an isomorphism of $g^{-1}(U)$ to U in some nonempty open subset $U \subseteq Y$. We can choose D' to be a general ample divisor in Y', such that D' intersects the proper transformation of f' inside U. Then D, the image of D', is a big divisor in Y, a subvariety of codimension 2 in X, and $f'(\mathbb{P}^1) \cap D \neq \emptyset$ for a general deformation f' of f. This contradicts the assumption that f is geometrically free. Hence f is weakly free. □

Weak freeness and geometric freeness are generalizations of usual freeness if the curve passes through singularities. To consider weakly free rational curves or geometrically free rational curves, we think of them as general members in a certain family.

Example 3.3.4. Let X be a projective cone over a conic. Let T be the family of all lines through the vertex O. Then $l \in T$ is not free. However l is weakly free and geometrically free over O by construction.

4 Main problem and results

4.1 Main problem and results

Main problem: Let X be an FT variety, and $P_1, \ldots, P_r \in X$ be finitely many points (P_i may be singular). Is there a geometrically free rational curve $f : \mathbb{P}^1 \to X$ over $P_1, \ldots, P_r$?

Theorem 4.1.1 (Main Theorem)**.** *Let X be a projective toric variety. Let $P_1, \ldots, P_r$ be finitely many points in X (P_i may be singular). Then there is a geometrically free rational curve $f : \mathbb{P}^1 \to X$ over $P_i, 1 \leq i \leq r$. Moreover, f is free if all points P_i are smooth.*

Lemma 4.1.2 (Main Lemma)**.** *Let X be a projective toric variety. Let $P, Q \in X$ be two distinct points (P, Q may be singular). Let $S \subseteq X$ be a closed subvariety of codimension ≥ 2. Then there exists a weakly free rational curve on X over P, Q, disjoint from $S \setminus \{P, Q\}$.*

If all points P_i are smooth, and $S = \operatorname{Sing} X$, then we get the following corollary.

Corollary 4.1.3. *The smooth loci of projective toric varieties are strongly rationally connected.*

4.2 Examples

In this section, we consider rational curves in the classical sense. That is, a curve C is called a rational curve if C is birational to $\mathbb{P}^1$.

4.2.1 Rational curves in projective space

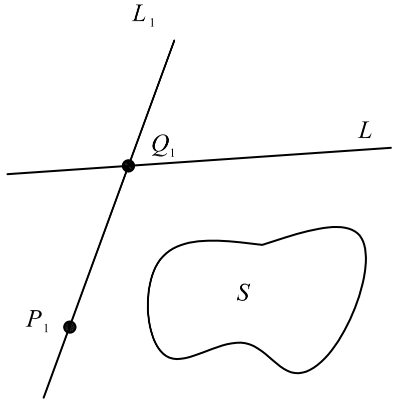
Theorem 4.2.1. *Let S be a subvariety in $\mathbb{P}^n$ of codimension ≥ 2 and $\{P_i\}_{i=1}^m$ be a set of m points in $\mathbb{P}^n \setminus S$. Then for any integer $d \geq m$, there exists a rational curve C of degree d, such that each $P_i \in C$ and $C \cap S = \emptyset$.*

Proof. **Case 1.** $n \geq 3$.

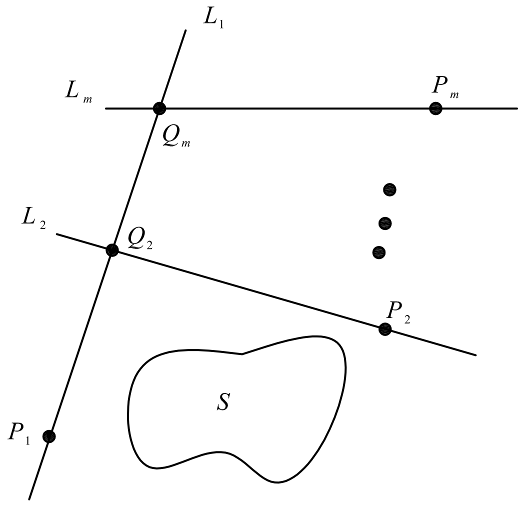
Step 1. There exist a tree of lines in $\mathbb{P}^n$, actually it is a comb with $m-1$ teeth, $T = \sum_{1 \leq i \leq m} L_i$ such that each L_i is a line and thus L_i is isomorphic to $\mathbb{P}^1$, $P_i \in L_i, 1 \leq i \leq m$, and $P_i \notin L_j$ for $i \neq j$. The tree has degree m.

Lemma 4.2.2. *(see [Musili01] 35.18) Let V be a closed subset of $\mathbb{P}^n$ of dimension $r < n$. Then there exists a linear subspace E_V of $\mathbb{P}^n$ of dimension $n-r-1$ such that $V \cap E_V = \emptyset$.*

We apply this lemma in our proof. Let $V = S \cup (\cup_{i \neq j} \overline{P_i P_j})$ which is a closed subset of dimension $r \leq n-2$. Then, by the lemma, there exists a linear subspace E_V with $\dim E_V = n - r - 1 \geq 1$ such that $E_V \cap V = \emptyset$. Then there exists a line L in E_V such that $L \cap V = \emptyset$ since E_V is a linear subspace. See Figure I.



I



II

Let C_{P_1} be the cone joining P_1 and all points on L, i.e., P_1 is the vertex. Then $C_{P_1} = \mathbb{P}^2$. Let $S_1 = S \cup_{i,j>1} \overline{P_i P_j}$, then we have $\dim S_1 \leq n-2$, and $C_{P_1} \cap S_1$ is a set of finitely many points. Indeed, if $\dim C_{P_1} \cap S_1 = \dim \mathbb{P}^2 \cap S \geq 1$, then there exists a curve $\widetilde{C} \subset C_{P_1} \cap S_1 = \mathbb{P}^2 \cap S_1$. By Bezout's theorem, $\widetilde{C} \cap L \neq \emptyset$ since $\widetilde{C}$ and L are in $\mathbb{P}^2$. We get $L \cap V \supset L \cap S_1 \neq \emptyset$. This contradicts the construction of L!

Since the line L has infinitely many points (here we can assume k is an algebraically closed field with characteristic 0), there exist points $Q_1 \in L$, such that the lines $\overline{P_1 Q_1} \cap S_1 = \emptyset$. Let $L_1 = \overline{P_1 Q_1}$.

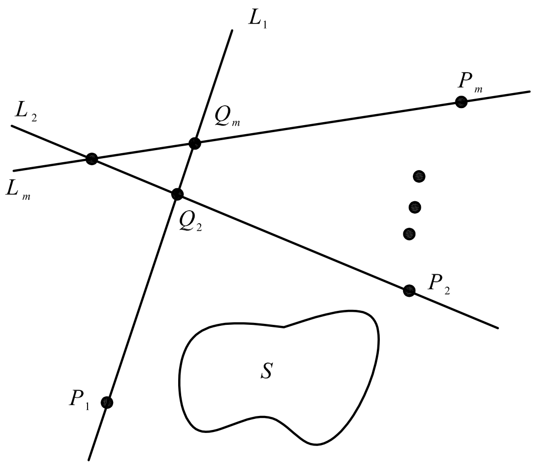
We will repeat our construction for L_1 above to get the other L_i (see figure II). Let $S_i = S \cup_{j,k>i} \overline{P_j P_k}$ and C_{P_i} be the cone joining P_i and line L_1 for $2 \leq i \leq m$. On the cone $C_{P_i}, 2 \leq i \leq m$, there exists $L_i = \overline{P_i Q_i}$ such that $L_i \cap L_1 = Q_i$ and $L_i \cap S_i = \emptyset, 2 \leq i \leq m$. From the construction, we have $L_i \cap L_j = \emptyset$ for $2 \leq i \neq j \leq m$. Indeed, if $L_{i_0} \cap L_{j_0}$ for some $2 \leq i_0 \neq j_0$, then P_{i_0}, P_{j_0} and L are in same plane $\mathbb{P}^2$. So $L_1 \cap \overline{P_{i_0} P_{j_0}} \neq \emptyset$. Contradiction!

Step 2. To get a rational curve C passing through $\{P_i\}$ by deformation with fix points $\{P_i\}$.

In step 1, we have constructed a tree of smooth rational curves passing through $\{P_i\}_{i=1}^m$ and disjoint with S. It is well known that $T_{\mathbb{P}^n}$ is ample, and so is $f^*T_{\mathbb{P}^n}|_{L_i} \cong T_{\mathbb{P}^n}|_{\mathbb{P}^1}$ since f is an embedding. Indeed, it's true for a finite morphism $g : Y \to X$ that g^*T_X is ample if T_X is ample. Applying Lemma 3.2.12, there is a rational curve C passing through P_i for all i. Since the tree is deformed smoothly, we have $C \cap S = \emptyset$ and the degree of the deformed curves is invariant. So, $\deg C = m$.

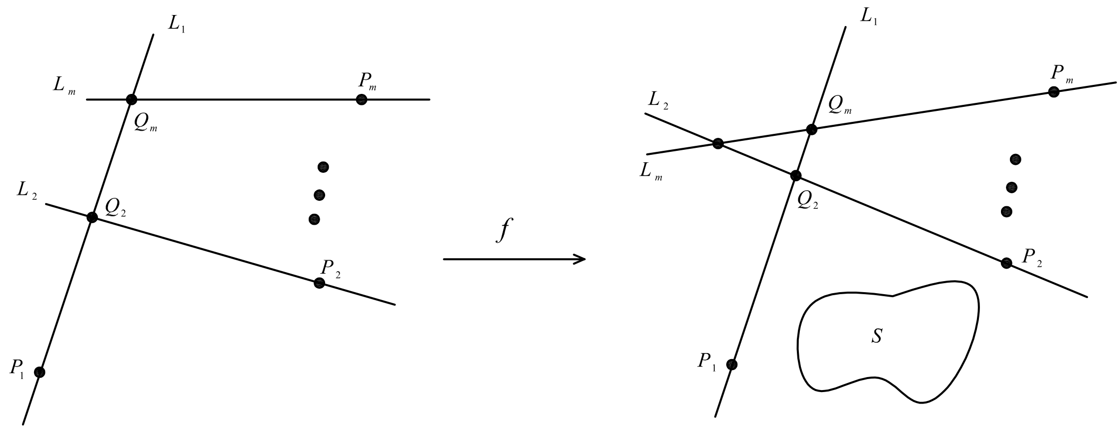
Case 2. $n = 2$.

In this case, S is a set of finitely many points. For $m \geq 3$, the above tree construction method doesn't work since every pair of lines intersects in $\mathbb{P}^2$ (see Figure III).



III

We could use a method similar to that to get lines L_i in Figure III such that the intersection points are at most nodes. Then we can do partial normalization or disjoint lines at the points $L_i \cap L_j, 2 \leq i \neq j \leq m$, to get a morphism f from a tree of lines T to $\mathbb{P}^2$ such that $f(T) = \sum L_i$ (see Figure IV). For any non-constant map from $\mathbb{P}^1$ into $\mathbb{P}^2$, the pull-back of the tangent bundle $T_{\mathbb{P}^2}$ to $\mathbb{P}^1$ is a sum of ample line bundles. Applying Lemma 3.2.12, we can construct a rational curve C passing through P_i for each i and disjoint from S.



IV

So far we have proved that there exists a rational curve of integral degree $d = m$ passing through m points $\{P_1, \ldots, P_m\}$ and disjoint from S. For $d > m$, we can add $d - m$ distinct points $\{P_{m+1}, \ldots, P_d\}$. Also we can assume $\{P_{m+1}, \ldots, P_d\} \cap \{P_1, \ldots, P_m\} = \emptyset$ since the ground field over which we are working is infinite. The same construction as above gives a degree d rational curve passing through $\{P_1, \ldots, P_m\}$ and disjoint from S. $\square$

Remark. In this case, $n = 1$, $S = \emptyset$, $C = \mathbb{P}^1$ and $d = 1$. However, there exists a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of any degree d such that $f(\mathbb{P}^1) = C$ passes through any point.

4.2.2 Rational curves on quotient spaces

In this section we consider the quotient space $\mathbb{P}^n/G$, where G is a finite group (possibly not commutative).

Theorem 4.2.3. *Let $\pi : \mathbb{P}^n \to X$ be a finite morphism, $H_X = \pi_* H_{\mathbb{P}^n}$, where $H_{\mathbb{P}^n}$ is a hyperplane in $\mathbb{P}^n$, S be a codimension ≥ 2 subvariety in X, and $\{P_i\}_{i=1}^m$ be a set of* *m points outside S. Then there exists a rational curve C of sufficiently large degree d with respect to H_X, such that each $P_i \in C$ and $C \cap S = \emptyset$. More precisely, we can take any $d \geq (m+1)\deg^s \pi$ with $\deg^s \pi | d$, where $\deg^s \pi$ denotes the separable degree of π.*

Remark. Due to the algebraic fact that every field extension E/k can be decomposed as $E \supset E_0 \supset k$, where E/E_0 is an inseparable extension and E_0/k is a separable extension, every finite morphism $\pi : \mathbb{P}^n \to X$ can be decomposed as

$$\mathbb{P}^n \xrightarrow{\alpha} X' \xrightarrow{\beta} X \ ,$$

that is $\pi = \beta\alpha$, β separable, α inseparable. Denote the separable degree $\deg^s \pi$ of π as $\deg \beta$ and the inseparable degree $\deg^i \pi$ of π as $\deg \alpha$.

In characteristic 0, $d \geq m \deg^s \pi$. But this may not be true if characteristic $p > 0$.

Proof. The inverse image $\pi^{-1}(S)$ has codimension ≥ 2 in $\mathbb{P}^n$. Choose a point Q_i in each fibre $\pi^{-1}(P_i)$, $1 \leq i \leq m$. By Theorem 4.2.1 there exists a rational curve $\widetilde{C} = f(\mathbb{P}^1)$ of any degree $\tilde{d} \geq m$, where $f : \mathbb{P}^1 \to \mathbb{P}^n$, passes through each point Q_i and is disjoint from $\pi^{-1}(S)$.

The image curve $C = \pi(\widetilde{C}) = \pi f(\mathbb{P}^1)$ is a rational curve in X disjoint from S. The degree of C with respect to H_X, can easily be computed:

$$H_X.C = H_X.\frac{1}{\deg \pi_{\widetilde{C}}}\pi_*\widetilde{C} = \frac{1}{\deg \pi_{\widetilde{C}}}\pi^* H_X.\widetilde{C} = \frac{\deg \pi}{\deg \pi_{\widetilde{C}}} H_{\mathbb{P}^n}.\widetilde{C} = \tilde{d} \cdot \frac{\deg \pi}{\deg \pi_{\widetilde{C}}},$$

where $\pi_{\widetilde{C}} = \pi|_{\widetilde{C}} : \widetilde{C} \to C$. The first equality comes from $C = \dfrac{1}{\deg \pi_{\widetilde{C}}}\pi_*\widetilde{C}$, the second one from the projection formula (see Theorem 1.2.8), that is, $H_X.\pi_*\widetilde{C} = \pi^* H_X.\widetilde{C}$, and the third one from $\pi^* H_X = \pi^*\pi_* H_{\mathbb{P}^n} = \deg \pi \cdot H_{\mathbb{P}^n}$.

Since $1 \leq \deg \pi_{\widetilde{C}} \leq \deg \pi$, we have the degree of C with respect to H_X, that is, $\tilde{d} \leq H_X.C \leq \tilde{d} \cdot \deg \pi$. Therefore there exists a sufficiently large degree curve C in X passing through the points $\{P_i\}$ disjoint from S.

Furthermore, we can construct a curve of degree $d = \deg^s \pi \cdot \tilde{d}$ for $\tilde{d} \geq m+1$. Indeed, for a generic point P_{m+1}, different from $\{P_i\}_{i=1}^m$, in X, we can choose a point Q_{m+1} in the fibre $\pi^{-1}(P_{m+1})$. Since P_{m+1} is generic, Q_{m+1} is generic and, for a general line L_{m+1} through Q_{m+1}, the map $\pi_{L_{m+1}} : L_{m+1} \to X$ has degree $\deg^i \pi$. We then switch Q_1 and Q_{m+1} and apply Theorem 4.2.1 to deform $\widetilde{C} \cup L_{m+1}$ to a rational curve $\widetilde{C}'$ passing through each $Q_i, 1 \leq i \leq m+1$. The curve $\widetilde{C}'$ has degree $m+1$ and its image $C' = \pi(\widetilde{C}')$ has degree $(m+1)\deg^s \pi$. □

Corollary 4.2.4. *Let G be a finite group, $\mathbb{P}^n/G$ be a quotient space, S be a codimension ≥ 2 subvariety in $\mathbb{P}^n/G$, and $\{P_i\}_{i=1}^m$ be a set of m points outside S. Then there exists a rational curve C with sufficiently large degree d, such that each $P_i \in C$ and $C \cap S = \emptyset$. More precisely, we can take $d \geq (m+1) \cdot \#G$ and $\#G | d$.*

Proof. The finite morphism $\pi : \mathbb{P}^n \to \mathbb{P}^n/G$ is separable and has degree $\#G$, so we can apply Theorem 4.2.3. □

Corollary 4.2.5. *Let S be a subvariety in a weighted projective space $\mathbb{P}^n(a_0, a_1, \ldots, a_n)$ of codimension ≥ 2 and $\{P_i\}, 1 \leq i \leq m$ be a set of m points outside S. Then there exists a rational curve C of sufficiently large degree d, such that each $P_i \in C$ and $C \cap S = \emptyset$. More precisely, we can take $d \geq (m+1) \cdot (b_0 \cdots b_n)$, where $\mathbb{P}^n(a_0, a_1, \ldots, a_n) \cong \mathbb{P}^n(b_0, b_1, \ldots, b_n)$ and $\mathbb{P}^n(b_0, b_1, \ldots, b_n)$ is a well formed projective space.*

Proof. It is well known that a weighted projective space $\mathbb{P}^n(a_0, a_1, \ldots, a_n)$ is isomorphic to a well formed weighted projective space $\mathbb{P}^n(b_0, b_1, \ldots, b_n)$ and $\mathbb{P}^n(b_0, b_1, \ldots, b_n)$

is a quotient $\mathbb{P}^n/G$ for the finite group $G = \mathbb{Z}_{b_0} \times \mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_n}$(see [Dolgachev82] 1.2.2 or [Fulton93] page 36, Exercise b)). Then by Corollary 4.2.4 we can get the estimate because $\#G = b_0 \cdots b_n$. □

Lemma 4.2.6. *Let X be a complete $\mathbb{Q}$-factorial toric variety with Picard number one. Then there exists a weighted projective space X' and a finite toric morphism $\pi : X' \to X$.*

Proof. Let Δ be the fan associated to the toric variety X. Here $\dim X = n$. Let $v_1, \cdots, v_{n+1}$ be the one dimensional cones in Δ. If the v_i's span the lattice N, then X is a weighted projective space, so there is nothing to prove. If not, then we take a sublattice N' spanned by the v_i's in N. If we consider Δ in N', then we obtain a weighted projective space X'. The natural inclusion $N' \to N$ induces a finite toric morphism $X' \to X$. □

Corollary 4.2.7. *Let S be a subvariety of X of codimension ≥ 2, where X is a complete $\mathbb{Q}$-factorial toric variety with Picard number $\rho(X) = 1$, and $\{P_i\}, 1 \leq i \leq m$ be a set of m points outside S. Then there exists a rational curve C of sufficiently large degree d, such that each $P_i \in C$ and $C \cap S = \emptyset$.*

Proof. If $\dim X = n$, then by Lemma 4.2.6 there exists a finite morphism π : $\mathbb{P}(a_0, \ldots, a_n) \to X$ from a weighted projective space to a toric variety with $\rho(X) = 1$. Thus there exists a finite morphism $\mathbb{P}^n \to X$. The corollary comes from Theorem 4.2.3. □

4.3 Proof of Main Lemma

Main Lemma 4.1.2 is the special weak case of Main Theorem 4.1.1 with $r = 2$ by Proposition 3.3.3.

Proof. Step 1. After $\mathbb{Q}$-factorization, we can assume that X is a projective $\mathbb{Q}$-factorial toric variety.

Indeed, for any toric variety X, there exists a small projective toric extraction $q : X' \to X$ such that X' is $\mathbb{Q}$-factorial. Take points $P', Q' \in X'$ such that $q(P') = P$ and $q(Q') = Q$. The inverse image $S' = q^{-1}S$ is a closed subvariety of codimension ≥ 2, because q is small.

By Lemma 4.3.5 below, a weakly free rational curve $f' : \mathbb{P}^1 \to X'$ over $P', Q' \in X'$, disjoint from $S' \setminus \{P', Q'\}$ gives a weakly free rational curve $f = q \circ f' : \mathbb{P}^1 \to X$ over $P, Q \in X$, disjoint from $S \setminus \{P, Q\}$.

Step 2. A weakly free rational curve can move from any smooth variety of codimension ≥ 2 in the following sense:

Lemma 4.3.1. *Let X be a projective normal variety. Let $P, Q \in X$ be two points (possibly singular), and S be a closed subvariety of codimension ≥ 2. Let $O_1, \ldots, O_s$ be all irreducible components of Sing X. Let $f : \mathbb{P}^1 \to X$ be a sufficiently general weakly free rational curve over P, Q. Suppose $f(\mathbb{P}^1)$ intersects $O_1 \setminus \{P, Q\}, \ldots, O_{s'} \setminus \{P, Q\}$, and is disjoint from $O_{s'+1} \setminus \{P, Q\}, \ldots, O_s \setminus \{P, Q\}$. Then there exists a weakly free rational curve f' over $\{P, Q\}$, such that $f'(\mathbb{P}^1)$ is disjoint from $((S \setminus \text{Sing } X) \cup O_{s'+1} \cup \ldots \cup O_s) \setminus \{P, Q\}$. Moreover, for any fixed closed subvariety Z of X, if $f(\mathbb{P}^1) \cap (Z \setminus \{P, Q\}) = \emptyset$, then $f'(\mathbb{P}^1) \cap (Z \setminus \{P, Q\}) = \emptyset$.*

Lemma 4.3.2. *Let T be an irreducible family of rational curves on X, and $O_1, \ldots, O_s \in X$ be s proper irreducible subvarieties in X. Then*

i) there exist $s', 0 \leq s' \leq s$, subvarieties among $\{O_j\}$ (after renumbering we assume they are $O_1, \ldots, O_{s'}$) and

ii) a general member of T intersects $O_1, \ldots, O_{s'}$, and is disjoint from $O_{s'+1}, \ldots, O_s$.

Proof of Lemma 4.3.2. Let $F_i = \{t \in T | f_t(\mathbb{P}^1) \cap O_i\}$ for each $1 \le i \le s$.

$$\begin{array}{ccc} \mathbb{P}^1 \times T & \xrightarrow{\text{ev}} & X \ , \\ {\scriptstyle p}\downarrow & & \\ T & & \end{array}$$

where ev: $\mathbb{P}^1 \times T \to X$ is the evaluation morphism and $p : \mathbb{P}^1 \times T \to T$ is the projection.

$F_i = p(\text{ev}^{-1}(O_i))$ for each i. Since O_i is an irreducible variety, it is constructible (see §1.2 for the definition of constructible sets). So $\text{ev}^{-1}(O_i)$ is a constructible set (see [Ha77] Chap II. Exercise 3.18). Since p is a morphism, $F_i = p(\text{ev}^{-1}(O_i))$ is constructible for each i (see [Ha77] Chap II. Exercise 3.19 (Theorem of Chevalley)).

After renumbering, we can assume $\dim F_1 = \dim F_2 = \cdots = \dim F_{s'} = \dim X$, for $0 \le s' \le s$, and $\dim F_j < \dim X$ for $s' + 1 \le j \le s$.

For each i, let

$$U_i = \begin{cases} F_i^\circ & \dim F_i = \dim X \\ X \setminus \overline{F_i} & \dim F_i < \dim X \end{cases}$$

where F_i° is the interior of F_i, and $\overline{F_i}$ is the closure of F_i. Each U_i is open and dense in the irreducible space T. Hence for a sufficiently general member t, that is, $t \in \cap_{i=1}^{s} U_i$, $f_t(\mathbb{P}^1)$ intersects $O_1, \dots, O_{s'}$, and is disjoint from $O_{s'+1}, \dots, O_s$. □

In the proof of Lemma 4.3.1, we need the following lemma about resolutions.

Lemma 4.3.3. *Let X be a projective normal variety. Let $P_1, \dots, P_r \in X$ be r points. Let $f : \mathbb{P}^1 \to X$ be a sufficiently general weakly free rational curve over $P_1, \dots, P_r$. Then there exists a resolution $\pi : \tilde{X} \to X$, such that*

1) $\pi^{-1}(\text{Sing } X \cup \{P_i\})$ is a divisor with simple normal crossing;

2) $\pi^{-1}(P_i) \subseteq \pi^{-1}(\text{Sing } X \cup \{P_i\})$ is a divisor for each point P_i;

3) $\pi : \tilde{X} \to X$ is an isomorphism over $X \setminus (\text{Sing } X \cup \{P_i\})$;

4) every point in $\tilde{f}(\mathbb{P}^1) \cap \pi^{-1}(\text{Sing } X \cup \{P_i\})$ *belongs to one irreducible component of* $\pi^{-1}(\text{Sing } X \cup \{P_i\})$*, where* $\tilde{f} : \mathbb{P}^1 \to \tilde{X}$ *is the strict transformation of a general deformation of* f*, which is a (weakly) free rational curve.*

Proof of Lemma 4.3.3. First, by Theorem 1.2.5, there exists a resolution $\pi_1 : \tilde{X}_1 \to X$ such that

a) $\pi_1 : \tilde{X}_1 \to X$ is an isomorphism over $X \setminus S_1$, where $S_1 = \text{Sing } X \cup \{P_i\}$;

b) $\pi_1^{-1}(P_i) \subseteq \pi_1^{-1}(S_1)$ is a divisor with simple normal crossing for each point P_i;

c) $\pi_1^{-1}(S_1)$ is a divisor with simple normal crossing.

Therefore, $\pi_1 : \tilde{X}_1 \to X$ is a resolution satisfying 1), 2) and 3).

By Lemma 4.3.4, there is a resolution $\pi_2 : \tilde{X} \to \tilde{X}_1$, such that every point of $\tilde{f}(\mathbb{P}^1)$ belongs to at most one component of the exceptional divisor of $\pi = \pi_1\pi_2 : \tilde{X} \to X$, where $\tilde{f}$ is the strict transformation under π of a general deformation f' of f.

Hence $\pi : \tilde{X} \to X$ is a resolution satisfying 1), 2), 3) and 4). □

Lemma 4.3.4. *Let* X *be a smooth projective variety (over* $\mathbb{C}$*). Let* $D \subseteq X$ *be an SNC divisor, and* $C \subseteq X$ *be a curve. Then there exists a resolution* $\pi : \tilde{X} \to X$*, such that*

1) π *is an isomorphism over* $X \setminus D$*;*

2) $\tilde{D} = \pi^{-1}D$ *is an SNC divisor;*

3) every point in $\tilde{C} \cap \tilde{D}$ *belongs to one irreducible component of* $\tilde{D}$*, where* $\tilde{C}$ *is the strict transformation of* C *under* π*.*

Proof of Lemma 4.3.4. First, notice that there are finitely many points in $C \cap D$, but they cannot be fixed for different C. Let $D_1, \ldots, D_r$ be all prime divisors of D.

If there is an n-point (see Definition 1.2.3) $P \in C \cap D$ with $n \geq 2$, denote $D^P = \cap_{P \in D_i} D_i$, where the intersection runs through all prime divisors $D_i, 1 \leq i \leq r$

of D such that $P \in D_i$. Let Z be the closure of $\cup_P D^P$, where the union runs through all n-points P in $C \cap D$ with $n \geq 2$. Let $\pi_1 : X_1 \to X$ be the blow up over Z. Then $\pi_1^* D = \sum \pi_1^{-1} D + \sum_j e_{1j} E_{1j}$, where E_{1j} is the π_1-exceptional prime divisor, and e_{1j} is the multiplicity of E_{1j}. If there is any point P' in $\pi_1^{-1}(C) \cap \pi_1^* D$ such that it is an n-point with $n \geq 2$, then we repeat the same construction for D^P, Z and get $D^{P'}, Z_1$. Let $\pi_2 : X_2 \to X_1$ be the blow up over Z_1. The process of such repeated blowing up will be terminated at some blowup $\pi_t : X_t \to X_{t-1}$. In the end, any point in $(\pi_t^{-1} \cdots \pi_1^{-1})(C) \cap (\pi_t^* \cdots \pi_1^*)(D)$ belongs to one prime divisor. For general C, π_t is independent of C.

Indeed, for each blow up, we have

$$(\pi_k^* \cdots \pi_1^*) D = (\pi_k^{-1} \cdots \pi_1^{-1}) D + \sum_{i=1}^{k} \sum_j e_{ij} E_{ij},$$

where E_{ij} is the π_i-exceptional prime divisor and e_{ij} is the multiplicity of E_{ij}. Moreover, we have $e_{ij} \geq i+1$ for each i, j. Indeed, this can be proved by induction. If $k = 1$, then $\pi_1 : X_1 \to X$ is the blow up over Z (see the construction above). So

$$\pi_1^* D = \pi_1^{-1} D + \sum_j e_{1j} E_{1j},$$

where the summation of j runs through all n-points P with $n \geq 2$. For an n-point $P \in C \cap D$ with $n \geq 2$, after renumbering, we can assume that $P \in D^P = D_1 \cap \cdots \cap D_n$. Then $\pi^{-1} D^P$ is a π_1-exceptional prime divisor, say E_{11}. Then

$$\pi_1^* D = (\pi_1^{-1} D_1 + \cdots + \pi_1^{-1} D_n + n E_{11}) + \cdots .$$

So $e_{11} = n \geq 2$. Now suppose that $(\pi_k^* \cdots \pi_1^*) D = (\pi_k^{-1} \cdots \pi_1^{-1}) D + \sum_{i=1}^k \sum_j e_{ij} E_{ij}$, and $e_{ij} \geq i+1$ for each i, j. π_{k+1} is the blow up of X_k over Z_k, where Z_k is constructed as Z and Z_1 above. Notice that if $P \in (\pi_k^{-1} \cdots \pi_1^{-1}) C \cap (\pi_k^* \cdots \pi_1^*) D$ is a n-point with $n \geq 2$, then $P \in E_{kj}$ for some j. The multiplicity $e_{k+1,j}$ of a π_{k+1}-exceptional prime divisor $E_{k+1,j}$ comes from $e_{kj'}$ for some j' and another prime divisor of $\pi_k^* \cdots \pi_1^* D$. So $e_{k+1,j} \geq e_{kj'} + 1 \geq k + 2$ by the induction hypothesis.

Suppose $t \geq m + 1$, where $m = C.D$ for general curve C in our family. Let $\pi = \pi_1 \cdots \pi_m$. Then by construction, $\pi^{-1}C.E_{m+1,1} \geq 1$, $\pi^{-1}C.E_{ij} \geq 0$ for each $i = 1, \ldots, m$ and $\pi^{-1}C.\pi^{-1}D_k \geq 0$ for each $k = 1, \ldots, r$, where $\pi^{-1}C$ is the strict birational transformation of C. So

$$\pi^{-1}C.\pi^* D = \pi^{-1}C. \left(\pi^{-1}D + \sum_{i=1}^{m+1} \sum_j e_{ij} E_{ij} \right) \geq e_{m+1,1}(\pi^{-1}C.E_{m+1,1}) \geq e_{m+1,1} \geq m+2.$$

On the other hand, by the projection formula (see Theorem 1.2.8), $\pi^{-1}C.\pi^* D = C.D = m$. Contradiction! $\square$

Proof of Lemma 4.3.1. By Lemma 4.3.3, there is a resolution $\pi : \tilde{X} \to X$, such that

1) $\pi^{-1}(\text{Sing } X \cup \{P, Q\})$ is an SNC divisor;

2) $\pi^{-1}(P)$ and $\pi^{-1}(Q)$ are SNC divisors;

3) $\pi : \tilde{X} \to X$ is an isomorphism over $X \setminus (\text{Sing } X \cup \{P, Q\})$;

4) every point in $\tilde{f}(\mathbb{P}^1) \cap \pi^{-1}(\text{Sing X} \cup \{P, Q\})$ belongs to only one irreducible component of $\pi^{-1}(\text{Sing } X \cup \{P, Q\})$, where $\tilde{f} : \mathbb{P}^1 \to \tilde{X}$ is the strict transformation of a general deformation of f, which is a weakly free rational curve. Moreover, we can assume $\tilde{f}$ is free by Theorem 3.2.6. (Here we need the assumption that the ground field is of characteristic 0.)

A general deformation $\tilde{f}'$ of $\tilde{f}$ is free, and is disjoint from $(S \setminus \text{Sing } X) \setminus \pi^{-1}\{P, Q\}$, since $\tilde{f}$ is free on $\tilde{X}$, and $(S \setminus \text{Sing } X) \setminus \pi^{-1}\{P, Q\}$ is a smooth subvariety of codimension ≥ 2. On the other hand, every point in $\tilde{f}(\mathbb{P}^1) \cap \pi^{-1}(\text{Sing } X \cup \{P, Q\})$ belongs to only one irreducible component of the divisor $\pi^{-1}(\text{Sing } X \cup \{P, Q\})$, and $\tilde{f}(\mathbb{P}^1)$ is disjoint from $\pi^{-1}(O_{s'+1} \setminus \{P, Q\}), \ldots, \pi^{-1}(O_s \setminus \{P, Q\})$. So a small deformation $\tilde{f}''$ of $\tilde{f}$ is disjoint from $\pi^{-1}(O_{s'+1} \setminus \{P, Q\}), \ldots, \pi^{-1}(O_s \setminus \{P, Q\})$, and every point in $\tilde{f}''(\mathbb{P}^1) \cap$

$\pi^{-1}(\text{Sing } X \cup \{P, Q\})$ belongs to only one irreducible component of $\pi^{-1}(\text{Sing } X \cup \{P, Q\})$. Therefore, we can assume $\tilde{f}''(\mathbb{P}^1)$ intersects open subsets in divisors $\pi^{-1}(P)$ and $\pi^{-1}(Q)$, disjoint from the closure of $((S \setminus \text{Sing } X) \setminus \pi^{-1}\{P, Q\}) \cup \pi^{-1}(O_{s'+1} \setminus \{P, Q\}) \cup \cdots \cup \pi^{-1}(O_s \setminus \{P, Q\})$. So $f'' = \pi \tilde{f}'' : \mathbb{P}^1 \to X$ is a weakly free rational curve passing through P, Q, disjoint from $((S \setminus \text{Sing } X) \cup O_{s'+1} \cup \cdots \cup O_s) \setminus \{P, Q\}$.

For any fixed closed subvariety Z of X, if $f(\mathbb{P}^1) \cap (Z \setminus \{P, Q\}) = \emptyset$, then $\tilde{f}(\mathbb{P}^1)$ is disjoint from $\pi^{-1}(Z \setminus \{P, Q\})$. Since every point in $\tilde{f}(\mathbb{P}^1) \cap \pi^{-1}(\text{Sing } X \cup \{P, Q\})$ belongs to only one irreducible component of the divisor $\pi^{-1}(\text{Sing } X \cup \{P, Q\})$, the small deformation $\tilde{f}''$ of $\tilde{f}$ is disjoint from the closure of $\pi^{-1}(Z \setminus \{P, Q\})$. Hence $f'' = \pi \tilde{f}''$ is disjoint from $Z \setminus \{P, Q\}$.

f'' belongs to an irreducible family T'' of rational curves. $S''_P \subseteq \{(t, f_t''^{-1}(P)) \subseteq T'' \times \mathbb{P}^1\}$ is an irreducible multi-section. $p : T'' \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection and $T'_P = S''_P$. We can assume $p' = p|_{T'} : T'_P \to T''$ is a finite etále cover (otherwise, we can consider p over an open subset of T'').

$$\begin{array}{ccc} T'_P = S''_P & \hookrightarrow & T'' \times \mathbb{P}^1 \\ & \searrow^{p'} & \downarrow p \\ & & T'' \end{array} \qquad\qquad \begin{array}{ccc} T'' \times \mathbb{P}^1 & \xleftarrow{q} & (T'' \times \mathbb{P}^1) \times_{T''} T'_P \\ p \downarrow & & \downarrow q' \\ T'' & \xleftarrow{p'} & T'_P \end{array}$$

By definition of fiber product, there exists a section $s_P : T'_P \to Y$, where $Y = (T'' \times \mathbb{P}^1) \times_{T''} T'$.

$S''_Q \subseteq q^{-1}(\{(t, f_t''^{-1}(Q)) \subseteq T'' \times \mathbb{P}^1\})$ is an irreducible multi-section. $T' = S''_Q$. We can assume $T' \to T'_P$ is a finite etále cover. Repeating the same construction, there is a section $s_Q : T' \to Y'$, where $Y' = Y \times_{T'_P} T'$.

$$\begin{array}{ccc} Y & \longleftarrow & Y' = Y \times_{T'_P} T' \\ q' \downarrow & & \downarrow \\ T'_P & \longleftarrow & T' \end{array}$$

Now there are two sections s_Q and the pull back of the section s_P over T'. Notice that they are disjoint, i.e., $s_P \cap s_Q = \emptyset$. Since T' is a family of rational curves over two points P, Q. We can assume that these two sections are constant after automorphisms of $\mathbb{P}^1$, that is, $s_P = T' \times 0$ and $s_Q = T' \times \infty$, where $0, \infty \in \mathbb{P}^1$. Hence general members over T' are weakly free rational curves over P, Q, disjoint from $((S \setminus \mathrm{Sing}\ X) \cup O_{s'+1} \cup \cdots \cup O_s) \setminus \{P, Q\}$. (Remark. The method works for at most 3 points. But, if $\dim T \geq 4 +$ the number of points, the method also works.) $\square$

Step 3. We can reduce the proof of Main Lemma 4.1.2 to the case $S = I(X)$, where $I(X)$ denotes the union of orbits of X of codimension ≥ 2. Notice that Sing $X \subseteq I(X)$, since X is a toric variety.

Indeed, for any subvariety $S \subseteq X$ of codimension ≥ 2, suppose there is a sufficiently general weakly free rational curve $f : \mathbb{P}^1 \to X$ over $P, Q \in X$, disjoint from $I(X) \setminus \{P, Q\}$. Apply Lemma 4.3.1 to the subvariety S, and the weakly free rational curve f. Since Sing $X \subseteq I(X)$, $s' = 0$ in Lemma 4.3.1, that is, $f(\mathbb{P}^1)$ is disjoint from $O_1 \setminus \{P, Q\}, \ldots, O_s \setminus \{P, Q\}$. Then there exists a general deformation f' of f, which is a weakly free rational curve, such that $f'(\mathbb{P}^1)$ is disjoint from $((S \setminus \mathrm{Sing}\ X) \cup O_1 \cup \ldots \cup O_s) \setminus \{P, Q\} = ((S \setminus \mathrm{Sing}\ X) \cup \mathrm{Sing}\ X) \setminus \{P, Q\} = S \setminus \{P, Q\}$.

Step 4. Let $O_1, \ldots, O_{\tilde{s}}$ be all orbits in $I(X)$. Let $f : \mathbb{P}^1 \to X$ be a sufficiently general weakly free rational curve over $P, Q \in X$. By Lemma 4.3.2, we can assume that $f(\mathbb{P}^1)$ intersects with $O_1 \setminus \{P, Q\}, \ldots, O_{s'} \setminus \{P, Q\}$, and is disjoint from $O_{s'+1} \setminus \{P, Q\}, \ldots, O_{\tilde{s}} \setminus \{P, Q\}$ for some s'.

Notice that s' depends on the points P, Q and the variety X. However, since s' is bounded by $\tilde{s}$, and $\tilde{s}$ is independent of isogeny class of X, there exists an s' such that for any toric variety Y in the isogeny class of X, and two distinct points

$P, Q \in Y$, there exists an irreducible family $T_{s'}$ of rational curves on Y such that a general member $f_{s'}$ over $T_{s'}$ is weakly free over P, Q, intersects with at most s' orbits $O_1 \setminus \{P, Q\}, \ldots, O_{s'} \setminus \{P, Q\}$ among $I(X)$, and is disjoint from $O_{s'+1} \setminus \{P, Q\}, \ldots, O_{\tilde{s}} \setminus \{P, Q\}$. Furthermore, we can assume that $\dim O_1 \geq \dim O_2 \geq \cdots \geq \dim O_{s'} \geq \dim O_{s'+1} \geq \cdots \geq \dim O_{\tilde{s}}$.

Now we verify that there is a weakly free rational curve $f_{s'-1}$ over P, Q on X such that $f'(\mathbb{P}^1)$ intersects at most $O_1 \setminus \{P, Q\}, \ldots, O_{s'-1} \setminus \{P, Q\}$, and is disjoint from $O_{s'} \setminus \{P, Q\}, \ldots, O_{\tilde{s}} \setminus \{P, Q\}$.

Indeed, we fix a toric variety X. We have the following cases:

1) If $f_{s'}(\mathbb{P}^1)$ is disjoint from $O_{s'} \setminus \{P, Q\}$, then $f_{s'-1} = f_{s'}$ is a weakly free rational curve over P, Q, such that $f_{s'-1}(\mathbb{P}^1)$ intersects at most $s' - 1$ orbits $O_1 \setminus \{P, Q\}, \ldots, O_{s'-1} \setminus \{P, Q\}$, and is disjoint from $O_{s'} \setminus \{P, Q\}, \ldots, O_{\tilde{s}} \setminus \{P, Q\}$.

2) If $f_{s'}(\mathbb{P}^1)$ intersects $O_{s'} \setminus \{P, Q\}$, we can suppose that $O_{s'}$ is smooth. Indeed, consider the isogeny $\mu : Y \to X$ such that $O^Y_{s'} = \mu^{-1}(O_{s'})$ is smooth. If $O_{s'}$ is smooth, let $\mu = \mathrm{id}$. Since μ is surjective, there are two points $\bar{P}, \bar{Q} \in Y$ such that $\mu(\bar{P}) = P$ and $\mu(\bar{Q}) = Q$. Let $O^Y_i = \mu^{-1}(O_i), 1 \leq i \leq \tilde{s}$. If there is a weakly free rational curve $f'_{s'-1} : \mathbb{P}^1 \to Y$ over $\bar{P}, \bar{Q}$ such that $f'_{s'-1}(\mathbb{P}^1)$ intersects at most $O^Y_1 \setminus \{\bar{P}, \bar{Q}\}, \ldots, O^Y_{s'-1} \setminus \{\bar{P}, \bar{Q}\}$, and is disjoint from $O^Y_{s'} \setminus \{\bar{P}, \bar{Q}\}, \ldots, O^Y_{\tilde{s}} \setminus \{\bar{P}, \bar{Q}\}$, then by Lemma 4.3.5, its image $f_{s'-1} = \mu f'_{s'-1}$ is a weakly free rational curve over P, Q, intersecting at most $O_1 \setminus \{P, Q\}, \ldots, O_{s'-1} \setminus \{P, Q\}$, and is disjoint from $O_{s'} \setminus \{P, Q\}, \ldots, O_{\tilde{s}} \setminus \{P, Q\}$.

Notice that $\dim O_1 \geq \dim O_2 \geq \cdots \geq \dim O_{\tilde{s}}$, $Z = O_{s'+1} \cup O_{s'+2} \cdots \cup O_{\tilde{s}}$ is a closed subvariety of X and $f_{s'}(\mathbb{P}^1) \cap (Z \setminus \{P, Q\}) = \emptyset$. By Lemma 4.3.1, there is a weakly free rational curve $f_{s'-1}$ on X, such that $f_{s'-1}(\mathbb{P}^1)$ is disjoint from $(O_{s'} \cup Z) \setminus \{P, Q\} = (O_{s'} \setminus \{P, Q\}) \cup (O_{s'+1} \setminus \{P, Q\}) \cup \cdots \cup (O_{\tilde{s}} \setminus \{P, Q\})$.

Lemma 4.3.5. *Let X, X' be two complete varieties (over a field of characteristic 0), and $\mu : X' \to X$ be a dominant morphism, $\dim X > 0$. Then the image of a weakly free rational curve on X' is weakly free on X in the following sense:*

Let $P_1, P_2, \ldots, P_r \in \mu(X)$ be r distinct points, and $S \subseteq X$ be a closed subvariety. Let $S' = \mu^{-1}S$, and $P'_1, P'_2, \ldots, P'_r \in X'$ be points such that $\mu(P'_i) = P_i$ for $i = 1, \ldots, r$. If $f' : \mathbb{P}^1 \to X'$ is a weakly free rational curve over $P'_1, P'_2, \ldots, P'_r$, disjoint from $S' \setminus \{P'_1, P'_2, \ldots, P'_r\}$, then $f = \mu \circ f''$ is a weakly free rational curve on X over $P_1, P_2, \ldots, P_r$, disjoint from $S \setminus \{P_1, P_2, \ldots, P_r\}$, where $f'' \in T'$ is a general member and T' is the family of rational curves associated to f'.

Proof of Lemma 4.3.5. Since f' is weakly free, ev: $\mathbb{P}^1 \times T' \to X'$ is dominant, where T' is the family associated to f'. Since $\mu : X' \to X$ is dominant, ev: $\mathbb{P}^1 \times T \to X' \to X$ is dominant. Hence for generic $f'' \in T'$, $f = \mu \circ f''$ is a weakly free rational curve (i.e., a general curve wouldn't be contracted by μ and would be weakly free). □

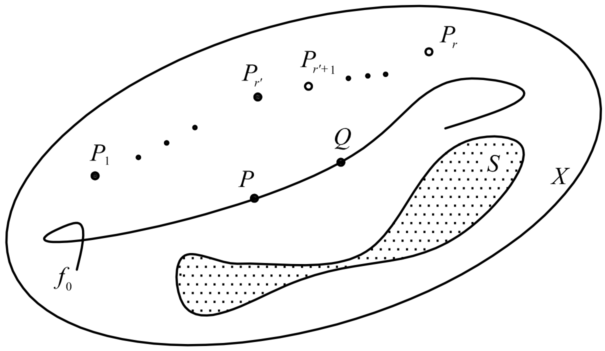
Step 5. By induction, there is a weakly free rational curve f_0 over P, Q on X, disjoint from $I(X) \setminus \{P, Q\}$. □

4.4 Proof of Main Theorem

Now we consider the general case, which has r points $P_i, 1 \leq i \leq r$.

Step 1. First we consider $S = \text{Sing } X$.

There is a free rational curve $f_0 : C_0 \cong \mathbb{P}^1 \to X$ disjoint from $\{P_i\} \cup S$. Indeed, we can apply Main Lemma 4.1.2 to the subvariety $\{P_i\} \cup S$ and any two smooth points $P, Q \notin \{P_i\} \cup S$ in X. So there is a weakly free rational curve $f_0 : C_0 \cong \mathbb{P}^1 \to X$ over P, Q, disjoint from $\{P_i\} \cup S$. Since $f_0(\mathbb{P}^1)$ is in the smooth locus of X, f_0 is free and disjoint from $\{P_i\} \cup S$.

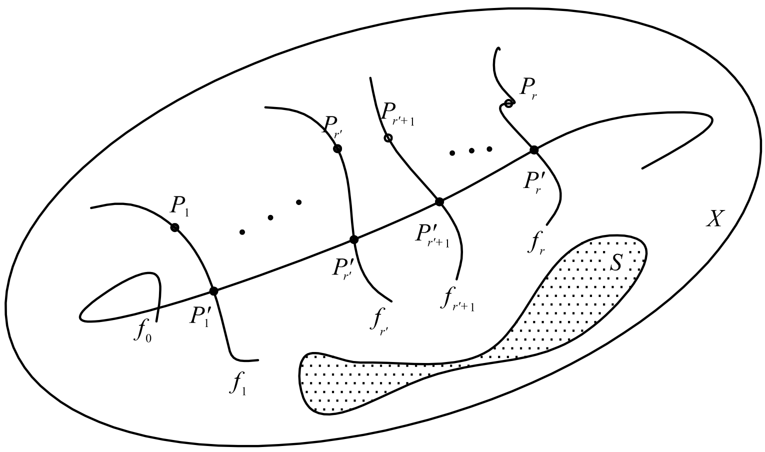


Let $\pi : X' \to X$ be the resolution in Lemma 4.3.3. We construct a tree of smooth rational curves C and a morphism $f : C \to X'$ as follows.

I. Assume that $P_1, \ldots, P_{r'}$ are smooth points for some r', $1 \le r' \le r$, and $P_{r'+1}, \ldots, P_r$ are singular points.

Choose points $t_1, \ldots, t_r \in C_0$, such that $P_i' = f_0(t_i) \in X$ are distinct.

Applying the Main Lemma 4.1.2 to $S = \operatorname{Sing} X$ and points P_i, P_i', there are weakly free rational curves $f_i : C_i \cong \mathbb{P}^1 \to X$ over P_i, P_i' for each $1 \le i \le r$, disjoint from $S \setminus \{P_i\}$.

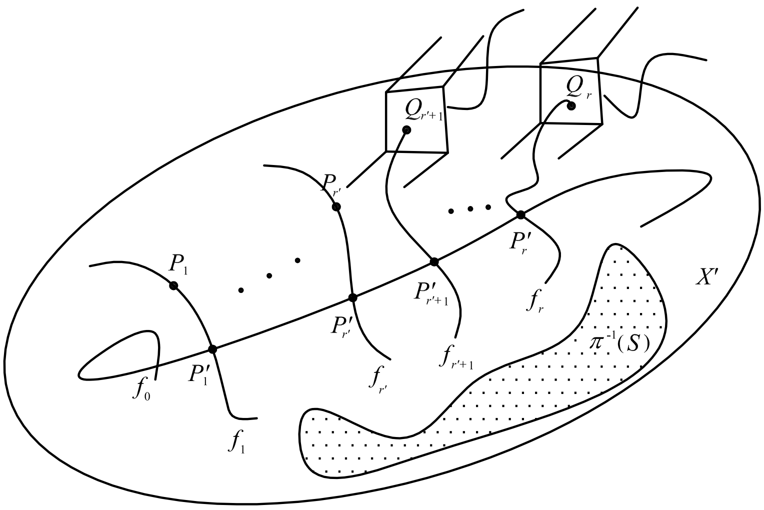


For each $1 \le i \le r'$, since P_i and P_i' are smooth points, $f_i(\mathbb{P}^1)$ is contained in the smooth locus of X. Therefore f_i are free for each $1 \le i \le r'$. We identify the free rational curve $f_i : C_i \cong \mathbb{P}^1 \to X$ birationally with a free rational curve

$f_i : C_i \cong \mathbb{P}^1 \to X'$. We also identify $P_i \in X$ with $P_i \in X', 1 \le i \le r'$, and $P'_i \in X$ with $P'_i \in X', 1 \le i \le r$. More precisely, $f_i(0_i) = P_i$, where $0_i \in C_i, 1 \le i \le r'$, and $f_i(\infty_i) = P'_i$ where $\infty_i \in C_i, 1 \le i \le r$.

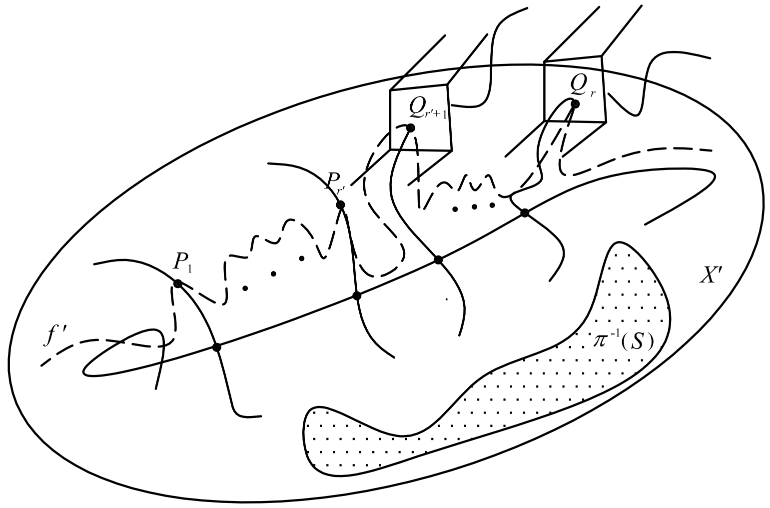
For each $r' + 1 \le j \le r$, P'_j is singular. Let $f'_j : C_j \cong \mathbb{P}^1 \to X'$ be the strict transformation of f_j. Since $\pi : X' \to X$ is after resolution satisfying all the conditions of Lemma 4.3.3, we assume that each point in $f'_j(C_j) \cap \pi^{-1}P_j$ belongs to at most one prime divisor of $\pi^{-1}P_j$ for $r'+1 \le j \le r$, disjoint from the closure of $\pi^{-1}(S \setminus \{P_i\}), \forall i$. Notice that f'_j may not be free over P'_j and Q_j. We can suppose that f_i is very free for $1 \le i \le r'$ and f'_j is very free for $r' + 1 \le j \le r$ by Theorem 3.2.6.

By construction of $f_i, 1 \le i \le r'$ and $f'_j, r' + 1 \le j \le r$, $f_i(C_i)$ and $f'_j(C_j)$ are disjoint from the closure of $\pi^{-1}(S \setminus \{P_1, \ldots, P_r\})$.



II. Gluing $\cup_{i=0}^r C_i$, we get a comb of smooth rational curves $C = \sum_{i=0}^r C_i$ (see Definition 3.2.9.) and a morphism $f : C \to X'$. We identify points $\infty_i \in C_i$ with $t_i \in C_0$ for each $1 \le i \le r$. Then we have a comb of smooth rational curves $C = \sum_{i=0}^r C_i$ and a morphism $f : C \to X'$ because $f_0(t_i) = f_i(\infty_i) = P'_i$. Notice that $f(C)$ is disjoint from the closure of $\pi^{-1}(S \setminus \{P_i\})$.

In the end, $f : C \to X'$ can be smoothed into a rational curve $f' : \mathbb{P}^1 \to X'$ such that f' is free over $P_i, 1 \leq i \leq r'$ and $Q_j, r'+1 \leq j \leq r$, and is disjoint from the closure of $\pi^{-1}(S \setminus \{P_i\})$ by Theorem 3.2.10.



Step 2. Now we consider any closed subvariety S of codimension ≥ 2.

By Step 1, there is a free rational curve $f' : \mathbb{P}^1 \to X'$ over $P_i, 1 \leq i \leq r'$ and $Q_j, r'+1 \leq j \leq r$, disjoint from the closure of $\pi^{-1}(\text{Sing } X \setminus \{P_i\})$. Since $S\setminus$ Sing X is a smooth subvariety of codimension ≥ 2 in X, we identify it with a smooth subvariety of codimension ≥ 2 in X'. Since f' is free over P_i, Q_j, $1 \leq i \leq r', r'+1 \leq j \leq r$ and disjoint from $\pi^{-1}(\text{Sing } X \setminus \{P_i\})$, a general deformation f'' of f' is free over P_i, Q_j, $1 \leq i \leq r', r'+1 \leq j \leq r$. Moreover, $f''(\mathbb{P}^1)$ is disjoint from $(S\setminus \text{ Sing } X) \setminus \{P_i\}$, or the closure of $\pi^{-1}(\text{Sing } X \setminus \{P_i\})$ by Theorem 3.2.5. Hence f'' is free over P_i, Q_j, $1 \leq i \leq r', r'+1 \leq j \leq r$ and disjoint from $\pi^{-1}(S \setminus \{P_i\})$. Therefore, $\pi f'$ is a geometrically free rational curve over $P_1, \ldots, P_r$ on X.

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