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1 Appetizers

Hint : Gauss lemma (cf problem set 3), ie

$$\forall (a,b,c) \in \mathbb{Z}^3, [(a|(bc)) \wedge (a \wedge b = 1)] \Rightarrow (a|c)$$

Exercise 1.

1. Compute the greatest common divisor d of 1482 and 1428.
2. Find some $u_0, v_0 \in \mathbb{Z}$ such that $d = 1482u_0 + 1428v_0$.
3. Find all the integers $u, v \in \mathbb{Z}$ such that $d = 1482u + 1428v$.

Exercise 2. Find all the integers $u, v \in \mathbb{Z}$ such $91u - 28v = 35$.

2 Main dish

Exercise 3. Let $(G_1, *_1)$ and $(G_2, *_2)$ be 2 groups and let f an homomorphism of groups from $(G_1, *_1)$ to $(G_2, *_2)$. Show that $\mathrm{Im}(G_1)$ is a subgroup of $(G_2, *_2)$.

Exercise 4. Let $(G_1, *_1)$ and $(G_2, *_2)$ be 2 groups and let f an homomorphism of groups from $(G_1, *_1)$ to $(G_2, *_2)$. Let H_2 a subgroup of $(G_2, *_2)$. Show that $f^{-1}(H_2)$ is a subgroup of $(G_1, *_1)$.

3 Dessert

Exercise 5.

1. Show that the set whose elements are the functions $f_{\varepsilon,b}\begin{cases} \mathbb{R}^2 \to & \mathbb{R}^2 \\ (x,y) \mapsto & (\varepsilon x + b, y) \end{cases}$ with $\varepsilon \in \{\pm 1\}, b \in \mathbb{R}$ equipped with the law $\circ$ is in fact a non-commutative (non abelian) group. Give a geometrical interpretation of this group.
2. Show that the set whose elements are the functions $f_{\varepsilon,a}\begin{cases} \mathbb{R}^2 \to & \mathbb{R}^2 \\ (x,y) \mapsto & (x + a, \varepsilon y) \end{cases}$ with $\varepsilon \in \{\pm 1\}, a \in \mathbb{R}$ equipped with the law $\circ$ is in fact a commutative (abelian) group. Give a geometrical interpretation of this group.

Exercise 6. We define two new operations on $\mathbb{R}^2$, $\oplus : \begin{cases} \mathbb{R}^2 \times \mathbb{R}^2 \to & \mathbb{R}^2 \\ ((a_1,b_1),(a_2,b_2)) \mapsto & (a_1 + a_2, b_1 + b_2) \end{cases}$ and $\otimes : \begin{cases} \mathbb{R}^2 \times \mathbb{R}^2 \to & \mathbb{R}^2 \\ ((a_1,b_1),(a_2,b_2)) \mapsto & (a_1 a_2, a_2 b_1 + a_1 b_2) \end{cases}$

1. Show that $(\mathbb{R}^2, \oplus)$ is a group.
2. Describe the set $(\mathbb{R}^2)^\times$ of the elements of $\mathbb{R}^2$ having an inverse for $\otimes$. (And so $((\mathbb{R}^2)^\times, \otimes)$is a group).
3. Show that the map $e : \begin{cases} \mathbb{R}^2 \to & (\mathbb{R}^2)^\times \\ (a,b) \mapsto & (\exp(a), b\exp(a)) \end{cases}$ is an homomorphism of groups.
4. Describe $\ker(e)$.