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Problem set 1 - Set theory, logic, proofs

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Skills to master:

- S1: Set theory, $\in$, $\subset$, $\cap$ and $\cup$.
- S2: Handling logical conjunctions.
- S3: Using truth tables.
- S4: Handling quantifiers.
- S5: Using the "necessary-sufficient conditions" method.
- S6: Using the induction principle.

Let us denote $\mathbb{N}^\star$ the set of positive integers.

Exercise 1 (S1)**.**
Let us denote $A = \{1, 2, 3, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.

1. Determine the set $A \cap B$.
2. Determine the set $A \cup B$.
3. Determine the set $\{a \in A / \ \exists b \in B, \ a = b - 1\}$.
4. Determine if the following assertions are true.
   (a) $2 \in A$
   (b) $B = \{n \in \mathbb{N}, \ n \text{ is odd}\}$
   (c) $A \in \mathbb{N}$
   (d) $B \subset A$

Exercise 2 (S2)**.**
Among the following propositions, determine which ones are equivalent to another one.

1. If it rains or snows, then the competition is cancelled.
2. If the competion is not cancelled, then it neither rains nor snows.
3. If the competion is cancelled, then it rains or it snows.
4. If it rains then the competion is cancelled and if it snows then the competion is cancelled.
5. If it neither snows nor rains, then the competion is not cancelled.

Exercise 3 (S2-S4).
We consider the following assertions :

a) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x + y > 0$

b) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x + y > 0$

c) $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x + y > 0$

d) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ y^2 > x$

For each assertion, say if it is true or false and then write its negation.

Exercise 4 (S2-S3).
We say that a proposition is a *tautology* when this proposition is always true.
Let A, B, C be three propositions.

1. Using a truth table, show that the proposition $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$ is a tautology.
2. Write the proposition $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$ only with $\vee, \wedge$ and $\neg$.
3. Using this decomposition show again (ie without using any truth table) that $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$ is a tautology.
4. Using a truth table, show that the proposition $(A \Rightarrow B) \Rightarrow (((B \Rightarrow C) \Rightarrow A) \Rightarrow B)$ is a tautology.

Exercise 5 (S5).
Using the "necessary-sufficient conditions" method:

1. Solve the equation $\sqrt{3 - x} = x - 2$ for $x \in \mathbb{R}$.
2. Prove that for any function $f : \mathbb{R} \to \mathbb{R}$, there exists an unique odd function $\omega : \mathbb{R} \to \mathbb{R}$ and an unique even function $\varepsilon : \mathbb{R} \to \mathbb{R}$ such that $f = \omega + \varepsilon$.

Exercise 6 (S6).

1. Prove by induction that $\forall n \in \mathbb{N}^\star, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
2. Can you find another proof of this fact without using the induction principle?
3. Prove by induction that $\forall n \in \mathbb{N}^\star, \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. Can you find another proof of this fact without using the induction principle?

Exercise 7 (S6).
This exercise is *hard*, therefore you should begin to work on small examples like $n = 2, 3, 4, \ldots$ to get an idea of what is going on.

1. Prove using the induction principle, for every $n \in \mathbb{N}^\star$ you need $n - 1$ breaks to split into small squares a chocolate bar containing n unit squares.
2. Poldevia is a very strange country : indeed in this country all roads are one-way only! Moreover in this country every pair of cities is connected by exactly one road. Prove using the induction principle that there exists a city (the capital of Poldevia) which can be reached from any other city in Poldevia either directly or by making step by just one other city.