Problem set 4 - Vocabulary of group theory

due: 4th October 2019

Exercise 1. Let $n \in \mathbb{N}^*$,

1. Show that $(GL_n(\mathbb{R}), \times)$ is a group.

2. Show that $SL_n(\mathbb{R})$ is a subgroup of $(GL_n(\mathbb{R}), \times)$.

Exercise 2. Let $(G, \ast)$ be a group. We denote $Aut(G)$ the set of all homomorphisms from $G$ to $G$ which are bijective, such homomorphisms are called automorphisms of $G$.

1. Show that $(Aut(G), \circ)$ is a group ($\circ$ being the composition of functions).

2. Let $g \in G$, we define $\varphi_g : \begin{cases} G \rightarrow G \\ x \mapsto g^{-1} \ast x \ast g \end{cases}$ Show that $\varphi_g \in Aut(G)$.

3. We denote $Inn(G) := \{ \varphi_g; g \in G \}$. Show that $Inn(G)$ is a subgroup of $(Aut(G), \circ)$.

Exercise 3. The aim of this exercise is to describe the set $Hom((\mathbb{Q}, +), (\mathbb{Q}^*_+, \times))$

1. Let us denote $I : \begin{cases} \mathbb{Q} \rightarrow \mathbb{Q}^*_+ \\ x \mapsto 1 \end{cases}$ Show that $I$ is a homomorphism from the group $(\mathbb{Q}, +)$ to the group $(\mathbb{Q}^*_+, \times)$ (ie $I \in Hom((\mathbb{Q}, +), (\mathbb{Q}^*_+, \times))$).

2. Let $f \in Hom((\mathbb{Q}, +), (\mathbb{Q}^*_+, \times))$ and $x \in \mathbb{Q}\setminus\{0\}$.

(a) Show that $\forall k \in \mathbb{N}^*, f(x) = (f(x/k))^k$.

(b) Show that $f(x) = 1$. (Hint : you may write the decomposition in prime numbers of the numerator and the denominator of $f(x)$ and study what the result of last question means on the exponents appearing in the decompositions in prime numbers of the numerator and the denominator of $f(x)$).

3. Show that $Hom((\mathbb{Q}, +), (\mathbb{Q}^*_+, \times)) = \{ I \}$

Exercise 4.

1. Explain (with words) how the group of symmetries of an isosceles triangle can be identified with $(\mathbb{Z}/2\mathbb{Z}, +_2)$.

2. Explain (with words) how the group of symmetries of an equilateral triangle can be identified with $(S_3, \circ)$ where $S_3$ denotes the set of of bijections from $[1, 3]$ to $[1, 3]$ and $\circ$ the composition of functions.

Exercise 5. Let $E$ be a non empty finite set and let $\ast$ an associative binary operation on $E$. Show that there exists an element $s \in E$ such that $s \ast s = s$. 