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Problem set 4 - Vocabulary of group theory

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Exercise 1. Let $n \in \mathbb{N}^{\star}$,

1. Show that $(\mathrm{GL}_n(\mathbb{R}), \times)$ is a group.
2. Show that $\mathrm{SL}_n(\mathbb{R})$ is a subgroup of $(\mathrm{GL}_n(\mathbb{R}), \times)$.

Exercise 2. Let $(G, *)$ be a group. We denote $\mathrm{Aut}(G)$ the set of all homomorphisms from G to G which are bijective, such homomorphisms are called *automorphisms* of G.

1. Show that $(\mathrm{Aut}(G), \circ)$ is a group ($\circ$ being the composition of functions).
2. Let $g \in G$, we define $\varphi_g : \begin{cases} G \to & G \\ x \mapsto & g^{-1} * x * g \end{cases}$ Show that $\varphi_g \in \mathrm{Aut}(G)$.
3. We denote $\mathrm{Inn}(G) := \{\varphi_g ; g \in G\}$. Show that $\mathrm{Inn}(G)$ is a subgroup of $(\mathrm{Aut}(G), \circ)$.

Exercise 3. The aim of this exercise is to describe the set $\mathrm{Hom}\big((\mathbb{Q}, +), (\mathbb{Q}_+^{\star}, \times)\big)$

1. Let us denote $\mathbb{1} \begin{cases} \mathbb{Q} \to & \mathbb{Q}_+^{\star} \\ x \mapsto & 1 \end{cases}$ Show that $\mathbb{1}$ is a homomorphism from the group $(\mathbb{Q}, +)$ to the group $(\mathbb{Q}_+^{\star}, \times)$ (ie $\mathbb{1} \in \mathrm{Hom}\big((\mathbb{Q}, +), (\mathbb{Q}_+^{\star}, \times)\big)$).
2. Let $f \in \mathrm{Hom}\big((\mathbb{Q}, +), (\mathbb{Q}_+^{\star}, \times)\big)$ and $x \in \mathbb{Q}\backslash\{0\}$.
   (a) Show that $\forall k \in \mathbb{N}^{\star}, f(x) = (f(x/k))^k$.
   (b) Show that $f(x) = 1$. (*Hint :* you may write the decomposition in prime numbers of the numerator and the denominator of $f(x)$ and study what the result of last question means on the exponents appearing in the decompositions in prime numbers of the numerator and the denominator of $f(x)$).
3. Show that $\mathrm{Hom}\big((\mathbb{Q}, +), (\mathbb{Q}_+^{\star}, \times)\big) = \{\mathbb{1}\}$

Exercise 4.

1. Explain (with words) how the group of symmetries of an isosceles triangle can be identified with $(\mathbb{Z}/2\mathbb{Z}, +_2)$.
2. Explain (with words) how the group of symmetries of an equilateral triangle can be identified with $(S_3, \circ)$ where S_3 denotes the set of of bijections from $[\![1, 3]\!]$ to $[\![1, 3]\!]$ and $\circ$ the composition of functions.

Exercise 5. Let E be a non empty finite set and let $*$ an associative binary operation on E. Show that there exists an element $s \in E$ such that $s * s = s$.