Automorphism Group of Blow-ups of $CP^k$

Turgay Bayraktar
Complex and Non-Archimedean Dynamics

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This is a joint work with Serge Cantat
Let $X$ be a compact Kähler manifold of dimension $k$.

Examples: Complex projective space $\mathbb{P}^k$, Complex torus $\mathbb{C}^k/\Lambda$, $K3$ surface etc..
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We denote the Chern map by

$$c_1 : Pic(X) \to H^2(X, \mathbb{Z}).$$

By a slight abuse of notation, $c_1(L)$ is also considered as an element of $H^2(X, \mathbb{R})$ (replacing $H^2(X, \mathbb{Z})$ by $H^2(X, \mathbb{Z}) \otimes \mathbb{R}$).
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The Néron-Severi group of $X$ is the image of the Chern map:

$$NS(X) = c_1(Pic(X)) \subset H^2(X, \mathbb{R}).$$

The Picard number is the rank of this free abelian group.
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It is a classical theorem that the group $\text{Aut}(X)$ is a complex Lie group, the Lie algebra of which is the algebra of holomorphic vector fields on $X$. 
Dynamical degrees

An automorphism $f \in Aut(X)$ induces a linear action

$$f^*_{p,q} : H^{p,q}(X, \mathbb{R}) \to H^{p,q}(X, \mathbb{R})$$

$$f^*_{p,q}\{\theta\} := \{f^*_{p,q}\theta\}$$

where $\{\theta\}$ denotes the class of the smooth $(p, q)$-form $\theta$ in $H^{p,q}(X, \mathbb{R})$. 
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For $f \in Aut(X)$ the $i^{th}$ dynamical degree

$$\lambda_i(f) = \text{spectral radius of } f^*_{i,i}.$$ 

that is $\lambda_i(f)$ is the largest (modulus of) eigenvalues of $f^*$ on $H^{i,i}(X, \mathbb{C})$. 
Let $f \in Aut(X)$ we consider the discrete dynamical system induced by $f$ i.e. we consider the iterates

$$f \circ f \circ \ldots \circ f$$
Topological entropy

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**Theorem (Gromov and Yomdin)**

Let $f$ be an automorphism of a compact Kähler manifold $X$. Then, the topological entropy of $f$ is given by

$$h_{\text{top}}(f) = \max_{0 \leq i \leq k} \log \lambda_i(f).$$

Therefore, $f$ has positive topological entropy if, and only if, $f^*$ has an eigenvalue larger than 1.
Some constraints on Aut(X)

The automorphism group $\text{Aut}(X)$ acts linearly on the cohomology of $X$ and this provides a homomorphism:

$$\text{Aut}(X) \to \text{GL}(H^*(X, \mathbb{Z}))$$

We denote the image of $\text{Aut}(X)$ under this homomorphism by $\text{Aut}(X)^*$. Theorem (Lieberman) The connected component of the identity $\text{Aut}_0(X)$ is contained in the kernel of this homomorphism. Thus, $\text{Aut}(X)$ has a finite number of connected components if and only if the group $\text{Aut}(X)^*$ is finite.
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**Theorem (Lieberman)**

The connected component of the identity

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has finite index in the kernel of this homomorphism.
Some constraints on Aut(X)

The automorphism group Aut(X) acts linearly on the cohomology of X and this provides a homomorphism:

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\text{Aut}(X) \to GL(H^\ast(X, \mathbb{Z}))
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f \to f^\ast
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We denote the image of Aut(X) under this homomorphism by Aut(X)*.

**Theorem (Lieberman)**

*The connected component of the identity*

\[\text{Aut}_0(X) \subset \text{Aut}(X)\]

*has finite index in the kernel of this homomorphism.*

Thus, Aut(X) has a finite number of connected components if and only if the group Aut(X)* is finite.
Compact complex surfaces

Introduction

Compact Kähler surfaces

Higher Dimensions

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Theorem (Cantat, Nagata)

If a compact complex surface $S$ admits an automorphism with positive entropy, then $S$ is Kähler and is obtained from the projective plane $\mathbb{P}^2$, a torus, a $K3$ surface or an Enriques surface, by a finite sequence of point blow-ups.

Examples of automorphisms with positive entropy are easily constructed on tori, $K3$ surfaces, or Enriques surfaces (see [Cantat] panorama survey).

On the other hand, examples of automorphisms with positive entropy on rational surfaces are harder to find. [Bedford and Kim, McMullen etc]. These examples are obtained from birational transformations $f$ of the plane by a finite sequence of blow-ups that resolves all indeterminacies of $f$ and $f^{-1}$ simultaneously.
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Automorphism group of blow-ups

Problem

*Does there exist an example of a birational transformation $f$ of the projective space $\mathbb{P}^k$, $k \geq 3$, which becomes an automorphism with positive entropy after a finite sequence of blow-ups?*
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We answer this question negatively in several cases and provide new criteria to prove that $\text{Aut}(X)^*$ is finite.
Main result

Theorem

Let $X_0$ be a smooth, connected, complex projective variety of dimension $k$ with Picard number 1 (resp. a compact Kähler manifold with $h^{1,1}(X_0) = 1$).

Let $m$ be a positive integer, and $\pi_i : X_{i+1} \to X_i$, $i = 0, \ldots, m - 1$, be a sequence of blow-ups of smooth irreducible subvarieties of dimension at most $r$.

If $k > 2r + 2$ then the group $\text{Aut}(X_m)^*$ is finite.
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If $k > 2r + 2$ then the group $\text{Aut}(X_m)^*$ is finite.

In particular,

- every automorphism of $X_m$ has zero topological entropy.
- $\text{Aut}(X)$ has finitely many connected components.
If $X_m$ is obtained from $X_0 = \mathbb{P}^k$, with $k \geq 3$, by blowing up a finite sequence of points then the topological entropy of all automorphisms of $X_m$ is equal to zero.
Examples

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- If $X_0$ is a smooth cubic hypersurface of $\mathbb{P}^{k+1}$ and $k \geq 3$ then the same statement applies.
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- If $X_0$ is a smooth cubic hypersurface of $\mathbb{P}^{k+1}$ and $k \geq 3$ then the same statement applies.

Thus, among all birational transformations of $X_0$, none of them lifts to an automorphism with positive entropy after a finite sequence of blow-ups of points (or of centers of dimension $< (k - 2)/2$).
Thanks!!