FIRST MIDTERM SOLUTIONS

1) (20 points, 2 parts)

a) Compute \( \lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x} \)

Solution:

\[
\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x} = \lim_{x \to 0} \frac{\frac{\sqrt{x^2 + 9} + 3}{\sqrt{x^2 + 9} + 3}}{x} = \lim_{x \to 0} \frac{\sqrt{x^2 + 9}}{\sqrt{x^2 + 9} + 3} = \frac{0}{6} = 0.
\]

b) Assign a value \( k \) so that the following function will be continuous

\[
f(x) = \begin{cases} 
  \frac{x^2 - x - 6}{x - 3}, & \text{if } x \neq 3 \\
  k, & \text{if } x = 3
\end{cases}
\]

Solution:

\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \frac{(x + 2)(x - 3)}{x - 3} = \lim_{x \to 3} (x + 2) = 5.
\]

Thus, if \( k = 5 \), then the resulting function is continuous.

2) (20 points, 2 parts)

a) Write down the precise definition of \( f' \) (i.e., the one involving limits).

Solution: If \( f \) is differentiable at \( x \) then

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

b) Using the definition of the derivative from part a), compute the equation of the tangent line to \( y = x^2 - 4x \) at the point \( x = 1 \).

Solution: If \( y = f(x) = x^2 - 4x \) then

\[
f'(x) = \lim_{h \to 0} \frac{(x + h)^2 - 4(x + h) - [x^2 - 4x]}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 4h}{h} = 2x - 4.
\]

Thus, the slope of the tangent line to the curve at \( x = 1 \) is \( f'(1) = 2 - 4 = -2 \). When \( x = 1, y = 1 - 4 = -3 \) on the curve, and so the slope of the tangent line through this point is

\[
y + 3 = -2(x - 1).
\]
3) (20 points, 2 parts)

Compute the derivatives of the following functions

a) \( x^{\sqrt{1+x}} \)

Solution:

\[
\frac{d}{dx} x^{\sqrt{1+x}} = \frac{d}{dx} e^{(\ln x)^{\sqrt{1+x}}} = e^{(\ln x)^{\sqrt{1+x}}} \frac{d}{dx} (\ln x)^{\sqrt{1+x}}
\]

\[
= x^{\sqrt{1+x}} \left[ \frac{1}{x} \sqrt{1+x} + \ln x \left( \frac{1}{2} \sqrt{1+x} \right) \right].
\]

b) \( \sin(\cos^3 x) \)

Solution:

\[
\frac{d}{dx} \sin(\cos^3 x) = \cos(\cos^3 x) \frac{d}{dx} \cos x^3 = \cos(\cos^3 x) (-\sin x^3 \frac{d}{dx} x^3)
\]

\[
= -3x^2 \sin x^3 \cos x^3 \cdot \frac{d}{dx} (\cos^3 x).
\]

4) (20 points) Consider the curve defined near \( P = (0, 1) \) by the equation

\[ y \cos x - e^{x^2} = 0. \]

Find the slope of its tangent line through \( P \).

Solution: We rewrite the equation as \( y \cos x = e^{x^2} \) and then differentiate both sides with respect to \( x \) to get

\[
\frac{dy}{dx} \cos x - y \sin x = e^{x^2} \frac{d}{dx} (x^2) = e^{x^2} \left( 2x \frac{dy}{dx} \right).
\]

If we plug in \( x = 0 \) and \( y = 1 \) to both sides, then the second term in the left and right vanish. Since \( \cos 0 = 1 \), this implies that \( dy/dx = 1 \) at \( P \), and so the slope of the tangent line through \( P \) is 1.

5) (20 points) A stone is dropped into a pond, the ripples forming concentric circles which expand. At what rate is the area of one of these circles increasing when the radius is 8 m and increasing at the rate of 0.5 m/s? (Hint: The area of a disk of radius \( r \) is \( \pi r^2 \).)

Solution: Let \( r \) be the radius of the circle. Then its area is \( A = \pi r^2 \). Both \( A \) and \( r \) are functions of \( t \). If we differentiate both sides with respect to \( t \), we get

\[
\frac{dA}{dt} = \frac{d}{dt} (\pi r^2) = 2\pi r \frac{dr}{dt}.
\]

by the chain rule. We are given that \( dr/dt = 1/2 m/s \) and \( r = 8m \). Thus, the rate of change of the area of the circle is

\[
\frac{dA}{dt} = 2\pi \cdot 8 \cdot \frac{1}{2} m^2/s = 8\pi m^2/s.
\]