1) a) Use the definition of the derivative to calculate $f'$ if $f(x) = \sqrt{6 + x}$.

**Solution:**

\[
f'(x) = \lim_{h \to 0} \frac{\sqrt{6 + x + h} - \sqrt{6 + x}}{h} = \lim_{h \to 0} \frac{\sqrt{6 + x + h} + \sqrt{6 + x}}{h} \frac{\sqrt{6 + x + h} - \sqrt{6 + x}}{h} = \lim_{h \to 0} \frac{h}{(\sqrt{6 + x + h} + \sqrt{6 + x})h} = \frac{1}{2\sqrt{6 + x}}.
\]

b) Find the equation of the tangent line to the graph of $f$ at $x = 3$.

**Solution:** $f'(3) = 1/2\sqrt{9} = 1/6$ and $f(3) = 3$, and so the equation is $y = \frac{1}{6}(x - 3) + 3$.

2) A 12-foot long ladder leans against a wall at an angle $\Phi$ with the horizontal as shown in the figure. The top of the ladder is $x$ feet above the ground. If the bottom of the ladder is is pushed toward the wall, find the rate at which $x$ changes with $\Phi$ when $\Phi = \pi/4$.

**Solution:** Note that $\sin \Phi = x/12$. Also, $x$ is a function of $\Phi$. If we differentiate we get

\[
\cos \Phi = \frac{d}{d\Phi} \sin \Phi = \frac{d}{d\Phi} \frac{x}{12} = \frac{1}{12} \frac{dx}{d\Phi}.
\]

Thus, $dx/d\Phi = 12 \cos \Phi$. Since $\cos \pi/4 = \sqrt{2}/2$, we have $dx/d\Phi = 6\sqrt{2}$ when $\Phi = \pi/4$. 

1
3) a) Find \((f \circ g)'(0)\) if \(f'(0) = 4, g(4) = 2, g'(4) = 4, g(0) = 0\) and \(g'(0) = 2\).

Solution:

\[
(f \circ g)'(0) = f'(g(0))g'(0) = f'(0)g'(0) = 4 \times 2 = 8.
\]

b) Find \(dy/dt\) if \(y = (\tan t)^t\).

Solution: \(y = e^{t \ln(\tan t)}\). Thus,

\[
\frac{dy}{dt} = e^{t \ln(\tan t)} \frac{d}{dt}(t \ln(\tan t))
\]

\[
= (\tan t)^t \left[ \ln(\tan t) + t \frac{d}{dt} \ln(\tan t) \right]
\]

\[
= (\tan t)^t \left[ \ln(\tan t) + t \frac{1}{\tan t} \frac{d}{dt} \tan t \right]
\]

\[
= (\tan t)^t \left[ \ln(\tan t) + t \frac{\cos t}{\sin t} \sec^2 t \right]
\]

\[
= (\tan t)^t \left[ \ln(\tan t) + t \sec t \csc t \right]
\]

4) Sketch the graph of \(y = 2x^4 - 4x^2 + 3\). Plot any stationary points or points of inflection.

Solution: We first note that

\[
y' = 8x^3 - 8x = 8x(x + 1)(x - 1).
\]

Therefore, the function has stationary points at \(x = 0, x = 1,\) and \(x = -1\). Since the derivative changes sign from \(-\) to \(+\) to \(-\) to \(+\) at these points, it is decreasing on \((-\infty, -1)\), increasing on \((-1, 0)\), decreasing on \((0, 1)\), and increasing on \((1, \infty)\). Also,

\[
y'' = 24x^2 - 8 = 24(x - 1/\sqrt{3})(x + 1/\sqrt{3}).
\]

So \(y\) is concave up on \((-\infty, -1/\sqrt{3})\), concave down on \((-1/\sqrt{3}, 1/\sqrt{3})\), and concave up on \((1/\sqrt{3}, \infty)\). It has inflection points at \(x = 1/\sqrt{3}\), and \(x = -1/\sqrt{3}\).

Its graph is below.
5) Evaluate the following limits

a) \( \lim_{x \to 1} \frac{x^2+5}{x+1} \)

\textbf{Solution:} \( \lim_{x \to 1} \frac{x^2+5}{x+1} = \frac{1+5}{1+1} = 3 \).

b) \( \lim_{x \to 0^+} x^{\sin x} \)

\textbf{Solution:} Note that \( x^{\sin x} = e^{\sin x \ln x} \). If we use L’Hopital, we find that

\[
\lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/\sin x} = \lim_{x \to 0^+} \frac{-1/\sin x}{-\cos x / \sin^2 x} = \lim_{x \to 0^+} \frac{-1}{\cos x} \cdot \frac{\sin^2 x}{x} = 0.
\]

Thus, \( \lim_{x \to 0^+} x^{\sin x} = e^{\lim_{x \to 0^+} \sin x \ln x} = e^0 = 1 \), since \( e^x \) is continuous.

6) Consider a population whose size at time \( t \) is \( N(t) \) and whose dynamics are given by the initial value problem

\[
\frac{dN}{dt} = e^{-t/2}
\]

with \( N(0) = 100 \).

a) Find \( N(t) \) by solving the initial value problem.

\textbf{Solution:} \( N(t) = \int e^{-t/2} \, dt = -2e^{-t/2} + C \). Since \( N(0) = 100 \), \( 100 = -2 + C \), and so \( C = 102 \). Thus, \( N(t) = 102 - 2e^{-t/2} \).

b) Compute the cumulative change in population size between \( t = 0 \) and \( t = 5 \).

\textbf{Solution:} \( N(5) - N(0) = -2e^{-5/2} + 2e^0 = 2(1 - e^{-5/2}) \).

7) (2 parts) Evaluate the following integrals

a) \( \int \frac{x^2+1}{\ln(x^2+1)} \, dx \)

\textbf{Solution:} Let \( w = x^2 + 1 \), then \( \frac{1}{2} \, dw = x \, dx \). Thus, the integral equals

\[
\frac{1}{2} \int \frac{dw}{w \ln w}.
\]

If we let \( u = \ln w \), then \( du = dw/w \). Thus, our integral equals

\[
\frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \ln(\ln(x^2 + 1)) + C.
\]
b) \( \int_1^2 x^2 \sqrt{x^3 + 2} \, dx \)

**Solution:** Let \( u = x^3 \), then \( \frac{1}{3} \, du = x^2 \, dx \). When \( x = 1 \), \( u = 1 \) and when \( x = 2 \), \( u = 8 \). Thus, the above integral equals

\[
\frac{1}{3} \int_1^8 \sqrt{u+2} \, du = \frac{1}{3} \left( (u+2)^{3/2} \right)_1^8 = \frac{2}{9} \left( 10^{3/2} - 3^{3/2} \right).
\]

8) (2 parts) Evaluate the following integrals

a) \( \int_0^3 x^2 e^{-x} \, dx \)

**Solution:** Let \( u = x^2 \) and \( dv = e^{-x} \, dx \). Then \( du = 2x \, dx \) and \( v = -e^{-x} \). The integral then equals

\[
-x^2 e^{-x}]_0^3 + 2 \int_0^3 x e^{-x} \, dx = -9e^{-3} + 2 \int_0^3 xe^{-x} \, dx.
\]

If we integrate by parts using \( x = u \), \( dv = e^{-x} \, dx \), we get

\[
\int_0^3 xe^{-x} \, dx = -xe^{-x}]_0^3 + \int_0^3 e^{-x} \, dx = -3e^{-3} - e^{-x}]_0^3 = -3e^{-3} - e^{-3} + 1.
\]

Combining the two steps gives

\[
\int_0^3 x^2 e^{-x} \, dx = -9e^{-3} - 6e^{-3} - 2e^{-3} + 2 = 2 - 17e^{-3}.
\]

b) \( \int_1^e \frac{dx}{x \sqrt{\ln x}} \)

**Solution:** Since \( \lim_{x \to 1^+} \frac{1}{x \sqrt{\ln x}} = +\infty \), this is an improper integral. To see if it exists, let us consider \( 0 < a < 1 \). If we let \( u = \ln x \) then \( du = 1/x \). When \( x = a \), \( u = \ln a \), and when \( x = 3 \), \( u = \ln e = 1 \). Thus,

\[
\int_a^e \frac{dx}{x \sqrt{\ln x}} = \int_{\ln a}^1 \frac{du}{\sqrt{u}} = 2u^{1/2}]_{\ln a}^1 = 2 - 2(\sqrt{\ln a}).
\]

This tends to 2 since \( \sqrt{\ln a} \) tends to 0 as \( a \to 1^+ \). Therefore the improper integral exists and equals 2.

9) (2 parts)

a) Find the best linear approximation of \( \sin x \) at \( x = 0 \).
Solution: \( \frac{d}{dx} \sin x = \cos x. \) Thus, if \( f(x) = \sin x, \) \( f'(0) = 1 \) and \( f(0) = 0. \) So the best approximation is \( L(x) = x. \)

b) Compute the Taylor polynomial of degree 5 about \( x = 0 \) for \( \sin x. \)

Solution: If \( f(x) = \sin x, \) \( f'(x) = \cos x, \) \( f''(x) = -\sin x, \) \( f'''(x) = -\cos x, \) \( f^{(4)}(x) = \sin x, \) and \( f^{(5)}(x) = \cos x. \) Thus, \( f(0) = 0, \) \( f'(0) = 1, \) \( f''(0) = 0, \) \( f'''(0) = -1, \) \( f^{(4)}(0) = 0, \) and \( f^{(5)}(0) = 1. \) So the Taylor polynomial is
\[
x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5.
\]

10) Compute the volume of the circular cone obtained by rotating the part of the line \( y = x \) between \( x = 0 \) and \( x = 2 \) around the \( x \)-axis.

Solution: Let \( 0 < x < 2. \) The area of the cross-section of the solid through \( x \) is \( \pi x^2. \) Thus, the volume is
\[
V = \int_0^2 \pi x^2 \, dx = \frac{1}{3} \pi x^3 \bigg|_0^2 = \frac{8\pi}{3}.
\]