Johns Hopkins University, Department of Mathematics
Calc 1 (Bio) - Spring 2013
Practice Midterm 2

Instructions: This exam has 4 pages (including a blank). No calculators, books, or notes are allowed. Show all work for all problems. No credit will be given for answers without work shown. If you do not have enough room in the space provided you may use additional paper. Be sure to clearly label each problem and attach them to the exam. You have 50 MINUTES.

Academic Honesty Certification

I certify that I have taken this exam without the aid of unauthorized people or objects.

Signature: Nick Marshburn
Date: 4/7/13

Name: Nick Marshburn
Section: 0

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1. (10 pts) Find $f'(x)$ for the following functions:

(a) (5 pts) $f(x) = 2^x$

$$f'(x) = \ln 2 \cdot 2^x \cdot -\frac{1}{x^2}$$

(b) (5 pts) $f(x) = x^{\ln x}$

\[
\begin{align*}
\ln f(x) &= \ln (x^{\ln x}) \\
\ln f(x) &= \ln x \cdot \ln x \\
\ln f(x) &= (\ln x)^2 \\
\frac{d}{dx} \ln f(x) &= \frac{d}{dx} (\ln x)^2 \\
\frac{f'(x)}{f(x)} &= 2(\ln x) \cdot \frac{1}{x} \\
f'(x) &= 2(\ln x) \cdot \frac{1}{x} \cdot f(x) = \frac{2 \ln x \cdot x \ln x}{x}
\end{align*}
\]
2. (5 pts) Approximate the value of \( e^{1.01} \) using the tangent line approximation to \( f(x) = e^x \) at \( x = 1 \).

\[
\begin{align*}
  f'(x) &= e^x \\
  f'(1) &= e \\
  y - f(1) &= f'(1)(x - 1) \\
  y - e &= e(x - 1) \\
  y - e &= ex - e \\
  y &= ex \\
  \text{If } x = 1.01, \text{ then } y = 1.01e.
\end{align*}
\]

3. (5 pts) Suppose \( f(x) \) is continuous on the closed interval \([0, 1]\), differentiable on the open interval \((0, 1)\), and \( f(1) < f(0) \). Explain why the derivative of \( f(x) \) is negative at least one point in \((0, 1)\).

By the MVT, there exists some \( c \) in \((0, 1)\) such that

\[
f'(c) = \frac{f(1) - f(0)}{1 - 0}, \quad \text{since } f(1) < f(0), \quad f'(c) < 0.
\]
4. (10 pts) Consider rectangles in the first quadrant of the xy-plane inscribed within the unit circle. Show that the rectangle of this type with the largest area is a square.

Clearly, our rectangle with maximal area should have its lower-left vertex at (0,0) and its upper-right vertex on the unit circle. Let \( x \) be the base of the rectangle and let \( y \) be the height. Then \( y = \sqrt{1-x^2} \) and the area is

\[
A(x) = xy = x \sqrt{1-x^2},
\]

\[
A'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} (1-x^2)^{-1/2} = 0
\]

Multiply through by \((1-x^2)^{1/2}\):

\[
(1-x^2) - x^2 = 0
\]

\[
1 - 2x^2 = 0
\]

\[
x^2 = 1
\]

\[
x = \sqrt{\frac{1}{2}}
\]

Use the first derivative test to classify the critical point \( x = \sqrt{\frac{1}{2}} \):

\[
A' \quad \frac{+}{\sqrt{\frac{1}{2}}} \quad \frac{-}{\sqrt{\frac{1}{2}}}
\]

\( \therefore x = \sqrt{\frac{1}{2}} \) is a max

\[
y = \sqrt{1-x^2} = \sqrt{1-\frac{1}{2}} = \sqrt{\frac{1}{2}}
\]

\( \therefore x = y \)

\( \therefore \) the rectangle with largest area is a square.
5. (5 pts) Find an antiderivative $F(x)$ of $f(x) = e^{-2x} + \frac{3}{x}$ with $F(1) = 0$.

$$F(x) = -\frac{1}{2} e^{-2x} + 3 \ln |x| + C$$

Use $F(1) = 0$ to find $C$:

$$0 = F(1) = -\frac{1}{2} e^{-2} + 3 \ln |1| + C = -\frac{1}{2} e^{-2} + C$$

$$\therefore C = \frac{1}{2} e^{-2}$$

$$F(x) = -\frac{1}{2} e^{-2x} + 3 \ln |x| + \frac{1}{2} e^{-2}$$

6. (5 pts) Compute $\lim_{x \to 0^+} (x^8 \ln x)$.

0 as Case of L'Hôpital's Rule.

$$= \lim_{x \to 0^+} \frac{\ln x}{x^{-8}} = \lim_{x \to 0^+} \frac{1/x}{-8x^{-9}} = \lim_{x \to 0^+} \frac{-x^2}{8} = 0$$
7. (5 pts) Let \( f(x) = 3x^3 + 10x^4 - 5x + 2 \). Locate the regions where \( f(x) \) is concave up or concave down and find all inflection points.

\[
f'(x) = 15x^4 + 40x^3 - 5
\]

\[
f''(x) = 60x^3 + 120x^2 = 60x^2(x + 2)
\]

\[
f''(x) = 0 \quad \text{when} \quad x = 0, -2
\]

\[
f''(x)
\begin{array}{ccc}
- & + & + \\
-2 & & 0
\end{array}
\]

\[
\therefore 
\begin{align*}
\text{Concave up on} & \quad (-2, \infty) \\
\text{Concave down on} & \quad (-\infty, -2) \\
\text{Inflection point at} & \quad x = -2
\end{align*}
\]

8. (5 pts) Find all horizontal and vertical asymptotes of \( f(x) = \frac{1 + e^{-x}}{1 - e^{-x}} \).

**Vertical**

\[
f(x) \text{ DNE when } 1 - e^{-x} = 0. \text{ This occurs when } x = 0.
\]

Check:

\[
\lim_{x \to 0^+} \frac{1 + e^{-x}}{1 - e^{-x}} = \infty \quad \therefore \quad x = 0 \text{ is a vertical asymptote}
\]

**Horizontal**

\[
\lim_{x \to \infty} \frac{1 + e^{-x}}{1 - e^{-x}} = \frac{1}{1} = 1 \quad \therefore \quad x = 1 \text{ is a horizontal asymptote}
\]

\[
\lim_{x \to -\infty} \frac{1 + e^{-x}}{1 - e^{-x}} = \frac{\lim e^{-x}}{\lim e^{-x}} = \frac{\lim (-1)}{\lim (-1)} = -1
\]

\[\therefore \quad x = -1 \text{ is a horizontal asymptote}
\]

\[\uparrow \text{L'Hopital}\]