Math 110.106 Calculus I (Bio. and Soc. Sci.)

Solutions First Midterm

March 2, 2012

NAME: ________________________________________________

TA: ___________________  SECTION NUMBER: ______________

1. Do not open this exam until you are told to begin.

2. This exam has 6 pages including this cover. There are 5 questions.

3. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.

4. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>POINTS</th>
<th>SCORE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>

Instructor: José Manuel Gómez

Johns Hopkins University
1. (10 Points) Compute the following limits.

(a) (2 Points) \( \lim_{x \to 2^+} \frac{x^2 - 4}{x - 2} \)

Solution:
\[
\lim_{x \to 2^+} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^+} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2^+} (x + 2) = 4.
\]

(b) (3 Points) \( \lim_{x \to 3^-} \frac{5x}{(x - 3)^2} \)

Solution: Note that \((x - 3)^2 \to 0\) as \(x \to 3^-\) and \(\frac{5x}{(x-3)^2} \geq 0\) when \(x \to 3^-\). Therefore
\[
\lim_{x \to 3^-} \frac{5x}{(x - 3)^2} = +\infty
\]

(c) (2 Points) \( \lim_{x \to \infty} \frac{4x^5 - x^2}{1 + x - 3x^5} \)

Solution:
\[
\lim_{x \to \infty} \frac{4x^5 - x^2}{1 + x - 3x^5} = -\frac{4}{3}
\]
because this is the limit of a rational function whose numerator has degree 5 and whose denominator also has degree 5. Thus we can compute this limit by looking at the leading coefficients.

(d) (3 Points) \( \lim_{x \to 0} \frac{\sin^2(2x)}{x} \)

Solution:
\[
\lim_{x \to 0} \frac{\sin^2(2x)}{x} = \lim_{x \to 0} \left( \frac{\sin(2x)}{2x} \right)^2 (2 \sin(2x))
= \lim_{x \to 0} \left( \frac{\sin(2x)}{2x} \right) \left( \lim_{x \to 0} 2 \sin(2x) \right)
= (1)(0) = 0.
\]
2. (10 Points) Consider the function $f$ with domain $(0, \infty)$ defined by

$$f(x) = \begin{cases} \frac{\sqrt{x} - 1}{x - 1} & \text{if } x \neq 1, \ x > 0 \\ a & \text{if } x = 1 \end{cases}$$

Find the value of $a$ such that $f$ is continuous in its domain. Please explain your arguments.

**Solution:** For $x \neq 1$ and $x > 0$ the function $f$ is defined by $f(x) = \frac{\sqrt{x} - 1}{x - 1}$ and this function is continuous there. Thus the only possible number where the function can fail to be continuous is $x = 1$. Let’s investigate the continuity at this point. To start note that $f(1) = a$ and in particular it is defined. Also,

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}.$$ 

The function $f$ is continuous at $x = 1$ if and only if

$$\lim_{x \to 1} f(x) = f(1)$$

and this is the case only if $a = \frac{1}{2}$. Thus $f$ is continuous everywhere if $a = \frac{1}{2}$. 
3. (10 Points)

(a) (4 Points) Show that the equation \(x^3 - x^2 + x - 2 = 0\) has at least a solution in the interval \([0, 2]\).

Solution: Consider the function \(f(x) = x^3 - x^2 + x - 2\). This is a continuous function. Moreover,
\[
\begin{align*}
    f(0) &= -2 < 0 \\
    f(2) &= 8 - 4 + 2 - 2 = 4 > 0.
\end{align*}
\]

By the intermediate value theorem we must have \(f(x) = 0\) for some \(x\) in the interval \([0, 2]\); that is, the equation \(x^3 - x^2 + x - 2 = 0\) has at least a solution in the interval \([0, 2]\).

(b) (3 Points) Solve the equation \(4^x = 9^{x+3}\).

Solution: Applying \(\ln\) in both sides of this equation we get
\[
x \ln(4) = (x + 3) \ln(9).
\]

This is \(x(\ln(4) - \ln(9)) = 3 \ln(9)\). Solving for \(x\) we obtain
\[
x = \frac{3 \ln(9)}{\ln(4) - \ln(9)}.
\]

(c) (3 Points) Determine if the sequence \(a_n\) defined below is convergent or divergent. If it is convergent then compute its limit. Please explain your answer.

\[
a_n = 2^{-n} + \frac{n - 1}{n + 1}
\]

Solution: Note that \(\lim_{n \to \infty} 2^{-n} = \lim_{n \to \infty} \frac{1}{2^n} = 0\). Also \(\lim_{n \to \infty} \frac{n - 1}{n + 1} = 1\) as it is a rational function where both the numerator and denominator have degree 1 and both have leading coefficient 1. Thus
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(2^{-n} + \frac{n - 1}{n + 1}\right) = \lim_{n \to \infty} 2^{-n} + \lim_{n \to \infty} \frac{n - 1}{n + 1} = 0 + 1 = 1.
\]

In particular this sequence converges to 1.
4. (10 Points)

(a) (5 Points) Consider the functions \( f(x) \) and \( g(x) \) defined as follows

\[
f(x) = x^2 + 1, \quad x \geq 4
\]

\[
g(x) = \sqrt{x}.
\]

Find \( (f \circ g)(x) \) together with its domain.

**Solution:** We have

\[
(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1.
\]

To find the domain of \( f \circ g \) note that the domain of \( g \) is the set of numbers \( x \geq 0 \). Thus the domain of \( f \circ g \) is the set of numbers \( x \geq 0 \) for which \( g(x) \) is in the domain of \( f \); that is, we need to solve the inequality \( \sqrt{x} \geq 4 \). This is \( x \geq 16 \). Thus the domain of \( f \circ g \) is the set of numbers \( x \geq 16 \).

(b) (5 Points) Compute

\[
\lim_{x \to \infty} \frac{\cos(x)}{\sqrt{x}}.
\]

**Solution:** We know that \(-1 \leq \cos(x) \leq 1\). Since \( \frac{1}{\sqrt{x}} \geq 0 \) for \( x \geq 0 \) we can multiply this inequality by \( \frac{1}{\sqrt{x}} \geq 0 \) to obtain

\[
-\frac{1}{\sqrt{x}} \leq \frac{\cos(x)}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}.
\]

Also

\[
\lim_{x \to \infty} \left(-\frac{1}{\sqrt{x}}\right) = 0 = \lim_{x \to \infty} \left(\frac{1}{\sqrt{x}}\right)
\]

By the sandwich theorem it follows that

\[
\lim_{x \to \infty} \frac{\cos(x)}{\sqrt{x}} = 0.
\]
5. (10 Points) Suppose that initially you have a 100 grams of a certain radioactive isotope. After 30 years there are only 20 grams left of the isotope.

(a) (5 Points) Find an equation for \( Q(t) \), the amount left of this isotope in grams, after \( t \)-years.

**Solution:** Since the amount of a radioactive isotope is an example of exponential decay we have

\[
Q(t) = Q_0 a^t,
\]

where \( Q_0 = 100 \) is the initial amount and \( a > 0 \) is some number. After 30 years we have 20 grams of the isotope. Therefore \( Q(30) = 20 \) and thus \( 20 = 100 a^{30} \). Solving for \( a \) we obtain \( a = \left( \frac{1}{5} \right)^{1/30} \). We conclude that

\[
Q(t) = 100 \left( \frac{1}{5} \right)^{t/30}.
\]

(b) (5 Points) What is the half life of this isotope?

**Solution:** We need to find the time \( t \) for which \( Q(t) = 50 \); that is, we need to solve the equation

\[
50 = 100 \left( \frac{1}{5} \right)^{t/30}.
\]

This is \( \left( \frac{1}{5} \right)^{t/30} = \frac{1}{2} \). Applying \( \ln \) in both sided and simplifying we obtain

\[
t = \frac{30 \ln(1/2)}{\ln(1/5)} = \frac{30 \ln(2)}{\ln(5)}.
\]