Back to recursions:

\[ a_{n+1} = \frac{4}{n+1} \cdot \frac{2^{n+1}}{7^n}, \quad a_0 = 2 \]

\[ \left(2, \frac{8}{7}, \frac{32}{49}, \frac{128}{343}, \ldots \right) \]

Guess: \( a_n = \frac{4^n}{n} \), Both: this is correct.

\[ \lim_{n \to \infty} \frac{4^n}{n} = 1 \] (check!)

What if we failed to guess formula for \( a_n \)?

A hint strategy: A fixed point of \( f \) is \( a \in \mathbb{R} \) s.t.

\[ f(a) = a \]

Fact: If \( a_{n+1} = f(a_n) \) recursively defined & \( (a_n) \) converges to \( a \), then \( a \) must be a fixed pt of \( f \) (provided \( f \) is cts at \( a \)) (to be def later)

Above eq: \( f(x) = \frac{2x}{7} + \frac{3}{7} \)

Solve for fixed pt: \( a = \frac{2}{7} a + \frac{3}{7} \implies \frac{3}{7} a = \frac{3}{7} \implies a = 1 \)

Checks!

Next

\[ a_{n+1} = \sqrt{5a_n}, \quad a_0 = 2 \]

Assuming \( \lim_{n \to \infty} a_n \) exists, find it.

Soln: Solve for fixed pt: \( a = \sqrt{5a} \implies a^2 = 5a \)

\( \implies a = 0 \) or \( 5 \)

In Few So many points: \( a = 2, 3, 1.62, 3.976, 9.459, \ldots \) \( \implies \lim \) is 5

Rem: \( \lim \) of a seq is unique if exists.

So if \( f \) has >1 fixed pts, at most 1 can be \( \lim \).

Def: Let \( a_1, a_2, \ldots, a_n \in \mathbb{R} \). Then

\[ \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n \] Sigma notation

\[ \sum_{k=1}^{2} k = \frac{(2)(2+1)}{2} = 3 \]

\[ \sum_{k=2}^{2} \sin(k) = 0 \]

\[ \sum_{k=-2}^{0} \sin(k) = \sin(-2) + \sin(-1) + \sin(0) + \sin(1) + \sin(2) \]

\[ = 0 \]

Sec 3.1 Limits

except if possibly not out of c.

"Def" Let \( f \) be def on some open interval containing \( c \).

Say the limit of \( f(x) \) as \( x \) approaches \( c \) is \( L \) if \( f(x) \) becomes arbitrarily close (but not necessarily =) to \( L \) whenever \( x \) is suff close, but not =, to \( c \)

Write: \( \lim_{x \to c} f(x) = L \) or \( f(x) \to L \) as \( x \to c \)

Say \( f \) converges to \( L \) at \( c \). (L unique if exists)

If \( f \) does not converge at \( c \), say \( f \) diverges at \( c \).

Version in terms of seqs:

\[ \lim_{x \to c} f(x) = L \iff \text{for any seq } \{x_n\} \text{ s.t. } x_n \to c \text{ and } x_n \neq c \text{ for } n \to \infty \]

\[ \lim_{x_n \to c} f(x_n) = L \]

\[ \lim_{x \to c} f(x) = L \text{ from the right} \]

\[ \lim_{x \to c^-} f(x) = L \text{ from the left} \]

\[ f(x) = \begin{cases} 
    x^2, & x \neq 2 \\
    1, & x = 2 
\end{cases} \]

\[ \lim_{x \to 2^+} f(x) = 4 \] - val of \( f \) at \( 2 \) irrelevant

\[ \lim_{x \to 2^-} f(x) = \frac{3}{2} \]

\[ \lim_{x \to 0^+} f(x) = 1, \lim_{x \to 0^-} f(x) = 1, \lim_{x \to 0} f(x) \text{ DNE} \]

"Def" \( \lim_{x \to 0} f(x) = \infty \) means \( f \) becomes arb large as \( x \to c \)

\[ \lim_{x \to 0} \frac{1}{x} = \infty \]

\[ \lim_{x \to 0} f(x) = 0 \]

\[ \lim_{x \to 0} f(x) \text{ DNE} \]

\[ f(x) = \sin \left( \frac{\pi}{x} \right) \]

Looks like \( \lim_{x \to 0} f(x) \) DNE.