Problem 1. [10 points] Let \( f(x, y) := e^x \ln y \), defined on the set of \((x, y) \in \mathbb{R}^2\) such that \( y > 0 \). Compute the linearization of \( f \) at the point \((1, 2)\).

Solution. We have
\[
\begin{align*}
    f(1, 2) &= e \ln 2, \\
    f_x(1, 2) &= (e^x \ln y)|_{(x, y)=(1, 2)} = e \ln 2, \\
    f_y(1, 2) &= \left( \frac{e^x}{y} \right)|_{(x, y)=(1, 2)} = \frac{e}{2}.
\end{align*}
\]
Therefore the linearization \( L(x, y) \) at \((1, 2)\) is given by
\[
L(x, y) = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) = e \ln 2 + (e \ln 2)(x - 1) + \frac{e}{2}(y - 2). \tag*{□}
\]

Problem 2. [10 points] Let \( f(x, y) := 3x^2y + 5xy - y^3 \). Evaluate the directional derivative of \( f \) at \((-1, 1)\) with respect to the unit vector pointing in the direction of greatest increase of \( f \) at this point.

Solution. We have
\[
\nabla f(-1, 1) = \begin{bmatrix} 6xy + 5y \\ 3x^2 + 5x - 3y^2 \end{bmatrix}|_{(x, y)=(-1, 1)} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}.
\]
Since the gradient points in the direction of greatest increase of \( f \), normalizing it gives the desired unit vector,
\[
\mathbf{u} := \frac{1}{\sqrt{26}} \begin{bmatrix} -1 \\ -5 \end{bmatrix}.
\]
Therefore the desired directional derivative is
\[
D_{\mathbf{u}} f(-1, 1) = \nabla f(-1, 1) \cdot \mathbf{u} = \frac{26}{\sqrt{26}} = \sqrt{26}. \tag*{□}
\]

Problem 3. [10 points] Find the critical points of the function \( f(x, y) := 3x^2 - 6xy + y^2 \) on \( \mathbb{R}^2 \) and determine whether each is a local maximum, a local minimum, or a saddle point.

Solution. The critical points are the solutions to the equation
\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \nabla f(x, y) = \begin{bmatrix} 6x - 6y \\ -6x + 2y \end{bmatrix}.
\]
From the first row we obtain \( x = y \). Plugging this into the second row, we obtain \( 0 = -4x \). Hence \( x = y = 0 \), so that \((0, 0)\) is the only critical point. To complete the problem, we compute the quantity
\[
f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 6 \cdot 2 - (-6)^2 = -24.
\]
Since this is negative, we conclude that \((0, 0)\) is a saddle point. \(\square\)

Problem 4. [10 points] Use the method of Lagrange multipliers to find the maximum value attained by the function \( f(x, y) := 2x + 4y^2 \) on the unit circle \( \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \).
Solution. Let \( g(x, y) := x^2 + y^2 - 1 \), so that the domain of the function is defined by the constraint equation \( g(x, y) = 0 \). Note that this domain is closed and bounded and \( f \) is continuous, which implies that \( f \) indeed attains a maximum value. By the method of Langrange multipliers, this maximum must be attained at a point \((x, y)\) which is a solution to
\[
\nabla f(x, y) = \lambda \nabla g(x, y)
\]
for some scalar \( \lambda \). Multiplying the first row by \( y \) and the second row by \( x \) and subtracting, we obtain
\[
2y - 8xy = 0 \implies y(1 - 4x) = 0.
\]
Hence \( y = 0 \) or \( x = 1/4 \). If \( y = 0 \), then by the constraint equation \( x = \pm 1 \), and we compute the values
\[
f(1, 0) = 2, \quad f(-1, 0) = -2.
\]
If \( x = 1/4 \), then by the constraint equation \( y = \pm \sqrt{15}/4 \), and we compute the values
\[
f\left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right) = f\left(\frac{1}{4}, -\frac{\sqrt{15}}{4}\right) = \frac{2}{4} + \frac{4 \cdot 15}{16} = \frac{17}{4}.
\]
The largest of these values is \( f(1/4, \sqrt{15}/4) = f(1/4, -\sqrt{15}/4) = 17/4 \), which therefore is the maximum.

□

Problem 5. [10 points] Find the general solution to the first order linear system
\[
\frac{dx}{dt} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} x(t),
\]
and find all equilibria solutions and evaluate their stability.

Solution. Let
\[
A := \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.
\]
Then
\[
\tau := \text{tr } A = 1 + 2 = 3 \quad \text{and} \quad \Delta := \det A = 1 \cdot 2 - 2 \cdot 3 = -4.
\]
The eigenvalues of \( A \) are then given by
\[
\frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2},
\]
i.e. 4 and \(-1\).

To find an eigenvector for 4, we need a nontrivial solution to the system
\[
\begin{bmatrix} 1 - 4 & 2 \\ 3 & 2 - 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
By inspection, \([2, 3]\) is a solution. Similarly, to find an eigenvector for \(-1\), we need a nontrivial solution to
\[
\begin{bmatrix} 1 - (-1) & 2 \\ 3 & 2 - (-1) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
By inspection, \([1, -1]\) is a solution. Hence the general solution is
\[
x(t) = C_1 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},
\]
where \( C_1 \) and \( C_2 \) are arbitrary constants.

Since \( \det A \neq 0 \), the only equilibrium solution is \( \hat{x} = [\frac{3}{2}] \). Since the eigenvalues of \( A \) are real and have opposite signs, \( \hat{x} \) is an unstable saddle.

□

Problem 6. [10 points]
(a) An urn contains six blue and four black balls. How many ways are there to choose (without replacement) three balls such that all are of the same color?

(b) Paula, Cindy, Gloria, Anne, and Jenny have dinner at a round table. In how many ways can they sit around the table if Cindy wants to sit to the left of Paula?

Solution. (a) The solution is the sum of the number of ways to choose 3 blue balls and the number of ways to choose 3 black balls, which is

\[
\binom{6}{3} + \binom{4}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} + \frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} = 20 + 4 = 24
\]

(b) There are 5 choices of seats for Paula. Having seated her, there is only 1 possible choice for Cindy. Having seated her, there are then 3 choices of seats for Gloria, and then 2 choices for Anne, and then 1 for Jenny. In total we find

\[5 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = 30\] possibilities.

\[\square\]