Formulas

- The differential equation \( \frac{dy}{dx} = k(y - a) \) for \( k, a \in \mathbb{R} \) has general solution \( y(x) = Ce^{kx} + a \).
- The differential equation \( \frac{dy}{dx} = k(y - a)(y - b) \) for \( k, a, b \in \mathbb{R} \) has general solution

\[
  y(x) = \begin{cases} 
    a - \frac{1}{kx + C}, & a = b, \ y \neq a; \\
    a - bCe^{k(a-b)x} \frac{1}{1 - Ce^{k(a-b)x}}, & a \neq b, \ y \neq b.
  \end{cases}
\]

Problem 1. [3 + 7 points] Consider the differential equation \( \frac{dy}{dt} = 2y^2 - 8 \).

(a) Find the general solution \( y(t) \). (It is OK for your expression to exclude the equilibrium values of \( y \).)
(b) Find the equilibrium values of \( y \) and evaluate their stability.

Solution. (a) Since the given equation is \( \frac{dy}{dt} = 2y^2 - 8 = 2(y - 2)(y + 2) \), we simply take \( k = 2, a = 2, b = -2 \) in the formula for the general solution:

\[
y(t) = \frac{2 + 2Ce^{2(2+2)t}}{1 - Ce^{2(2+2)t}} = \frac{2 + 2Ce^{8t}}{1 - Ce^{8t}}
\]

(b) Let \( g(y) := 2y^2 - 8 \).

The equilibria are the values \( \hat{y} \) for which \( g(\hat{y}) = 0 \), which are the values \( a \) and \( b \) above, i.e. \( \pm 2 \). To evaluate the stability of these equilibria, we use the derivative

\[
\frac{dg}{dy}(y) = 4y.
\]

Then

\[
\frac{dg}{dy}(2) = 8 > 0 \implies \hat{y} = 2 \text{ is unstable}
\]

and

\[
\frac{dg}{dy}(-2) = -8 < 0 \implies \hat{y} = -2 \text{ is locally stable}.
\]

Problem 2. [10 points] Let \( a, b \in \mathbb{R} \) be constants, and consider the linear system

\[
4x + ay = -6 \\
x - 2y = b.
\]

For which values of \( a \) and \( b \) does the system have (i) infinitely many solutions and (ii) no solutions? For the values of \( a \) and \( b \) in (i), determine the solution set of the system.

Solution 1. There are three possibilities for the lines in \( \mathbb{R}^2 \) corresponding to the solution sets of each of the two equations: they intersect once (one solution), they are the same line (infinitely many solutions), or they are parallel but distinct (no solution). Note that if \( a = 0 \), then the first equation yields \( x = -3/2 \), and substituting this into the second equation yields \( y = 3/4 - b/2 \) — a unique solution. So to find the values of
a and b corresponding to (i) or (ii), it is safe to assume that $a \neq 0$. Rewritten in slope-intercept form, the two equations then become

$$
y = -\frac{4}{a} x - \frac{6}{a},$$

$$
y = \frac{1}{2} x - \frac{b}{2}.
$$

(i) For there to be infinitely many solutions, then, we must have an equality of slopes

$$\frac{-4}{a} = \frac{1}{2} \implies a = -8.$$ 

and an equality of constant terms

$$\frac{-b}{2} = -\frac{6}{a} = -\frac{6}{-8} = \frac{3}{4} \implies b = -\frac{3}{2}.$$ 

(ii) For there to be no solutions, the slopes must be equal to have parallel lines, so we still have $a = -8$. But the lines must have different $y$-intercepts, and thus $b \neq -3/2$.

Finally, the solution set in (i) corresponds to the line that both equations lie on, that is,

$$\begin{array}{l}
\{(x, y) \in \mathbb{R}^2 \mid y = \frac{1}{2} x + \frac{3}{4} \} = \left\{ \left( x, \frac{1}{2} x + \frac{3}{4} \right) \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\} \quad \square
\end{array}$$

Solution 2. Let

$$
A := \begin{bmatrix} 4 & a \\ 1 & -2 \end{bmatrix}, \quad X := \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad B := \begin{bmatrix} -6 \\ b \end{bmatrix}.
$$

Then the linear system can be expressed as the matrix equation

$$AX = B.$$ 

When the matrix $A$ is invertible, there is a unique solution $X = A^{-1}B$. Hence for there to be infinitely many or no solutions, $A$ must be non-invertible, which holds if and only if

$$0 = \det A = -8 - a \iff a = -8.$$ 

So let us take $a = -8$ for the rest of the problem. Then we row reduce the augmented matrix of the linear system,

$$
\begin{bmatrix} 4 & -8 & -6 \\ 1 & -2 & b \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - 4R_2} \begin{bmatrix} 0 & 0 & -6 - 4b \\ 1 & -2 & b \end{bmatrix}.
$$

This system is consistent if and only if $-6 - 4b = 0$, i.e., if and only if $b = -3/2$. For this value of $b$, we set $y = t$ for a parameter $t$, and we obtain from the second row in the last matrix that $x = -3/2 + 2y = -3/2 + 2t$. Thus (i) we find an infinite solution set exactly when $a = -8$ and $b = -3/2$,

$$\left\{ \begin{bmatrix} -3/2 + 2t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\};$$

and (ii) we find that there are no solutions exactly when $a = -8$ and $b \neq -3/2$. \square

Remark. Note that the presentations of the solution sets in (i) are different in Solutions 1 and 2, i.e., the sets are written down in different ways. But the sets themselves really are the same!

Problem 3. [10 points] Find the eigenvalues and all corresponding eigenvectors of the matrix $A = \begin{bmatrix} 2 & 4 \\ 3 & -2 \end{bmatrix}$.
Solution. The eigenvalues $\lambda$ are the solutions to $\det(A - \lambda I) = 0$:

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 4 \\ 3 & -2 - \lambda \end{bmatrix} = (2 - \lambda)(-2 - \lambda) - 4 \cdot 3 = \lambda^2 - 16 = (\lambda - 4)(\lambda + 4)$$

$$\implies \lambda = 4 \text{ or } \lambda = -4.$$

The eigenvectors for $\lambda = 4$ are the nonzero solutions $[x_1 \ x_2]$ to $(A - 4I)[x_1 \ x_2] = 0$. We express this linear system in terms of its augmented matrix,

$$\begin{bmatrix} -2 & 4 & 0 \\ 3 & -6 & 0 \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{-1}{2} \cdot R_1} \begin{bmatrix} -2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Setting $x_2 = t$ for a parameter $t$, we find that $x_1 = 2x_2 = 2t$. Thus the set of eigenvectors for $\lambda = 4$ is

$$\left\{ \begin{bmatrix} 2t \\ t \end{bmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\}.$$

Analogously, the eigenvectors for $\lambda = -4$ are the nonzero solutions $[x_1 \ x_2]$ to $(A + 4I)[x_1 \ x_2] = 0$. We express this linear system in terms of its augmented matrix,

$$\begin{bmatrix} 6 & 4 & 0 \\ 3 & 2 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2} \cdot \cdot \cdot \cdot R_2 \rightarrow R_1} \begin{bmatrix} 6 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Setting $x_2 = t$ for a parameter $t$, we find that $x_1 = -\frac{2}{3}x_2 = -\frac{2}{3}t$. Thus the set of eigenvectors for $\lambda = -4$ is

$$\left\{ \begin{bmatrix} -\frac{2}{3}t \\ t \end{bmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\}.$$

Problem 4. [10 points] Find all real numbers $a > 0$ for which the integral

$$\int_1^\infty \frac{1}{x^a} \, dx$$

converges. (HINT: Consider the cases $a = 1$ and $a \neq 1$ separately.)

Solution. First, for $a = 1$,

$$\int_1^\infty \frac{1}{x} \, dx = \lim_{z \to \infty} \int_1^z \frac{1}{x} \, dx = \lim_{z \to \infty} \ln |x| \bigg|_1^z = \lim_{z \to \infty} \ln |z| - 0 = \infty.$$

Hence the integral diverges when $a = 1$.

For $a \neq 1$,

$$\int_1^\infty \frac{1}{x^a} \, dx = \lim_{z \to \infty} \int_1^z \frac{1}{x^a} \, dx = \lim_{z \to \infty} x^{1-a} \bigg|_1^z = \frac{1}{1-a} \left( \lim_{z \to \infty} z^{1-a} \right) - \frac{1}{1-a}.$$

Now for $p > 0$, $\lim_{z \to \infty} z^p = \infty$, and for $p < 0$, $\lim_{z \to \infty} z^p = 0$. This corresponds to $1 - a > 0$ and $1 - a < 0$, respectively. So the above integral diverges when $a < 1$ and converges when $a > 1$. \qed

Problem 5. [3 + 7 points]

(a) If a function $f(x, y)$ is defined on some open disk around $(x_0, y_0)$, what does it mean for $f$ to be continuous at $(x_0, y_0)$?

(b) Is the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2 - y^2}, & (x, y) \neq (0, 0); \\ 0, & (x, y) = (0, 0) \end{cases}$$

continuous at $(0, 0)$? Justify your answer. (HINT: What is our usual way to probe a question like this?)
Solution. (a) The function $f$ is continuous at $(x_0, y_0)$ if $\lim_{(x,y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$.

(b) If we restrict $f$ to the line $y = mx$ for fixed $m \in \mathbb{R}$, we obtain

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{x \to 0} \frac{mx^2}{x^2 - m^4x^4} = \lim_{x \to 0} \frac{m}{1 + m^4x^2} = m.$$ 

Thus different values of $m$ (e.g., $m = 0$ and $m = 1$) give different values for this limit. We conclude that $f$ approaches different limits along different paths through $(0,0)$, and therefore $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist. Therefore $f$ is not continuous at $(0,0)$. \hfill \Box$