Lecture Notes for Math 104: Fall 2010 (Week 4)

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CHAPTER 1

Ninth and Tenth Lectures

In this Lecture I started to discuss complementary decompositions of vector spaces as well as projectors.

1. Orthogonal Projections

One of the key uses of inner products is that they allow one to decompose arbitrary vectors into orthogonal components. This often simplifies a problem substantially. We say this already when one has a basis but the idea is more general.

The basic idea: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthonormal set of vectors. For $\mathbf{w}$ an arbitrary vector we have that $\mathbf{v}_i^*\mathbf{w}$ is a scalar. If we write

$$r = \mathbf{w} - (\mathbf{v}_1^*\mathbf{w})\mathbf{v}_1 - (\mathbf{v}_2^*\mathbf{w})\mathbf{v}_2 - \ldots - (\mathbf{v}_k^*\mathbf{w})\mathbf{v}_k$$

Then it is straight forward to check that r is orthogonal to S that is we can write:

$$\mathbf{w} = r + (\mathbf{v}_1^*\mathbf{w})\mathbf{v}_1 + (\mathbf{v}_2^*\mathbf{w})\mathbf{v}_2 + \ldots + (\mathbf{v}_k^*\mathbf{w})\mathbf{v}_k = r + \sum_{i=1}^{k}(\mathbf{v}_i\mathbf{v}_i^*)\mathbf{w}$$

so all the summand vectors are orthogonal. Note the second equality just uses the fact that scalar multiplication can be commuted.

Notice that if the $\mathbf{v}_i$ form a basis then $r = 0$. That is we have expressed $\mathbf{w}$ in terms of the basis $\mathbf{v}_i$ in a relatively painless manner. This is one of the real powers of inner products and orthogonality.

I point out also that when I rewrote the expansion in terms of $(\mathbf{v}_i\mathbf{v}_i^*)\mathbf{w}$ I wasn't doing much mathematically but the interpretation is important. The poin this $\mathbf{v}_i\mathbf{v}_i^*$ is now a square matrix P_i that one can check preserves any vector $\lambda\mathbf{v}_i$ (i.e. $P_i(\lambda\mathbf{v}_i) = \lambda\mathbf{v}_i$ and has $Null(P_i)$ the space of vectors orthogonal to $\mathbf{v}_i$. That is P_i is the projection matrix onto $\mathbf{v}_i$. These are important special cases of a more general type of matrix that will be important for us.

Notice that philosphically the two expansions are different. The first we view $\mathbf{w}$ as coefficients $\mathbf{v}_i^*\mathbf{w}$ times the $\mathbf{v}_i$ plus some left over term r while in the second we view $\mathbf{w}$ as the sum of vectors $(\mathbf{v}_i\mathbf{v}_i^*)\mathbf{w}$ given by projecting plus some left over term r. projections

We come up with a more general concept of orthogonal projection if we let

$$P = \sum_{i=1}^{k}(\mathbf{v}_i\mathbf{v}_i^*)$$

then $\mathbf{w} = P\mathbf{w} + r$. Here P is $m \times m$ matrix with rank k. P gives projection onto the span of the $\mathbf{v}_i$. For instance if $k = 2$ this is projection onto a plane. We point out that $P^2 = P$ and $P^* = P$. I.e. P is *idempotent* and *hermitian*. We will return to this soon.

2. Sums of vector spaces

In order to get a better sense of what is going on with orthogonal projection we need some ideas about vector subspaces. It will also help to take a slightly more general point of view.

To that end let us suppose that we have two vector spaces $E_1, E_2 \subset \mathbb{C}^n$. We denote by $E_1 + E_2$ the vector space so that $\mathbf{w} \in E_1 + E_2$ when and only when there are $\mathbf{v}_1 \in E_1, \mathbf{v}_2 \in E_2$ so that $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$. It is straightforward to check that $E_1 + E_2$ is a vector space and I leave it as an excercise.

An important fact is that if $E_1 \cap E_2 = \{0\}$ then each $\mathbf{w} \in E_1 + E_2$ can be written *UNIQUELY* as $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in E_1$ and $\mathbf{v}_2 \in E_2$. To see this suppose that $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w} = \mathbf{v}_1' + \mathbf{v}_2'$ where $\mathbf{v}_i, \mathbf{v}_i' \in E_i$. By equating the two sides we have $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_1' + \mathbf{v}_2'$ that is $\mathbf{v}_1 - \mathbf{v}_1' = \mathbf{v}_2' - \mathbf{v}_2$ we denote the common value by $\mathbf{v}$. Notice the left hand side is in E_1 while the right hand side is in E_2 and so $\mathbf{v} \in E_1 \cap E_2$ and so $\mathbf{v} = 0$. In other words $\mathbf{v}_1 = \mathbf{v}_1'$ and $\mathbf{v}_2 = \mathbf{v}_2'$.

We will mostly be interested in situations where E_1 and E_2 span $\mathbb{C}^n$ that is $E_1 + E_2 = \mathbb{C}^n$ and $E_1 \cap E_2 = \{0\}$. In this case we say that E_1 and E_2 are *complimentary*. The idea here is now that any vector $\mathbf{w} \in \mathbb{C}^n$ can be written as $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_i \in E_i$.

For example: Let $\{\mathbf{b}_i\}$ be a basis of $\mathbb{C}^n$, $i = 1, \ldots, n$ if $E_1 = \textit{span}\,\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ and $E_2 = \textit{span}\,\{\mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ then E_1 and E_2 are complementary subspaces.

One important task is: Given two complementary vector spaces E_1, E_2 in $\mathbb{C}^n$ we know that for any $\mathbf{w} \in \mathbb{C}^n$ we have $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_i \in E_i$ and this decomposition is unique. The question is to what extent can we determine $\mathbf{v}_1$ from $\mathbf{w}$.

We claim that in fact there is a fairly straightforward answer to this question. Namely there is a $n \times n$ matrix P so that $\mathbf{v}_1 = P\mathbf{w}$. Such a P is called a *projector*. We will return to them in a bit.

Before disucssing projectors we wish to point out one final thing. If E_1 and E_2 are orthogonal subspaces in $\mathbb{C}^n$ and they span $\mathbb{C}^n$ then they are automatically complementary (any $\mathbf{v} \in E_1 \cap E_2$ would satisfy $\langle \mathbf{v}, \mathbf{v} \rangle = 0$). In this case E_1 and E_2 are said to be *orthogonal complements*. We've already seen that it is fairly straightforward to find the orthogonal projector in this case. Indeed, let us use our fact that we can find $\{\mathbf{q}_1, \ldots, \mathbf{q}_k\}$ an orthonormal basis of E_1 then as we've seen for any $\mathbf{w} \in \mathbb{C}^n$

$$\mathbf{w} = (\sum_{j=1}^{k} \mathbf{q}_j \mathbf{q}_j^*)\mathbf{w} + \mathbf{r}$$

where $\mathbf{r}$ is orthogonal to the $\mathbf{q}_i$ and hence to E_1 and so lies in E_2. In particular our projector in this case is $P = \sum_{j=1}^{k} \mathbf{q}_j \mathbf{q}_j^*$.

3. Projectors

As mentioned in the previous section for any pair of complementary spaces $E_1, E_2 \subset \mathbb{C}^n$ there are matrices P_1 and P_2 in $\mathbb{C}^{n \times n}$ so that $P_1\mathbf{w} \in E_1$ and $P_2\mathbf{w} \in E_2$ and $\mathbf{w} = P_1\mathbf{w} + P_2\mathbf{w}$. We then call the P_i *projectors*.

To see this we argue as follows. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{C}^n$. For each i the fact that E_1 and E_2 is orthogonal allows us to write

$$\mathbf{e}_i = \mathbf{a}_i + \mathbf{b}_i$$

so that $\mathbf{a}_i \in E_1$ and $\mathbf{b}_i \in E_2$ and the $\mathbf{a}_i$ and $\mathbf{b}_i$ are neccesarily unique. We then set:

$$P_1 = \left[\mathbf{a}_1 | \quad \cdots \quad | \mathbf{a}_n\right], P_2 = \left[\mathbf{b}_1 | \quad \cdots \quad | \mathbf{b}_n\right].$$

And claim that P_1 and P_2 are the desired matrices. To see this it suffices to show

LEMMA 3.1. *Let* $\mathbf{v} \in E_1$ *then* $P_1\mathbf{v} = \mathbf{v}$ *and* $P_2\mathbf{v} = 0$.

PROOF. Write $\mathbf{v} = \sum_{i=1}^n v_i\mathbf{e}_i = \sum_{i=1}^n v_i(\mathbf{a}_i + \mathbf{b}_i) = \sum_{i=1}^n v_i\mathbf{a}_i + \sum_{i=1}^n v_i\mathbf{b}_i$. Notice that the first summand is in E_1 while the second is in E_2. Since we can also write $\mathbf{v} = \mathbf{v} + 0$ where the first summand is in E_1 and the second is in E_2 by the uniquess of the decomposition (as E_1 and E_2 are complementary we have

$$\mathbf{v} = \sum_{i=1}^n v_i\mathbf{a}_i$$

and

$$0 = \sum_{i=1}^n v_i\mathbf{b}_i$$

One the other hand, $P_1\mathbf{v} = P_1\left(\sum_{i=1}^n v_i\mathbf{e}_i\right) = \sum_{i=1}^n v_i\mathbf{a}_i$ and $P_2\mathbf{v} = \sum_{i=1}^n v_i\mathbf{b}_i$. □

COROLLARY 3.2. *If* $\mathbf{w} \in \mathbb{C}^n$ *then* $P_1\mathbf{w} \in E_1$ *and* $P_2\mathbf{w} \in E_2$ *and* $\mathbf{w} = P_1\mathbf{w} + P_2\mathbf{w}$.

PROOF. The columns of P_1 are in E_1 so $R(E_1) \subset E_1$ and similarly $R(P_2) \subset E_2$. Now write $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ with $\mathbf{w}_1 \in E_1$ $\mathbf{w}_2 \in E_2$. We see that $P_1\mathbf{w} = P_1(\mathbf{w}_1 + \mathbf{w}_2) = P_1\mathbf{w}_1 + P_1\mathbf{w}_2 = \mathbf{w}_1$ by the proceeding lemma. Similarly, $P_2\mathbf{w} = \mathbf{w}_2$. □

Notice that this proof is not constructive so we don't have a good way to find P.

4. Projectors

It is useful to formalize the notion of a projector as a property inherent to a matrix. This allows us to more easily answer and manipulate questions about complementary subspaces. To that end, we say a $n \times n$ matrix P is an *oblique projector* (or just *projector*) if $P^2 = P$ (such a property is often called begin idempotent). Notice that Lemma 3.1 implies that the matrix P_1 is a projector in this sense.

Once you have a projector P, you can get a different projector $Q = I - P$ called the *complementary projector*. We check this as follows: $Q^2 = (I-P)^2 = I - P - P + P^2 = I - P = Q$. Note that P is then the complementary projector of Q. A useful fact is that $N(P) = R(Q)$ and $N(Q) = R(P)$. Check this: $\mathbf{w} = Q\mathbf{v} = \mathbf{v} - P\mathbf{v}$ then $P(\mathbf{v}) - P^2(\mathbf{v}) = 0$. So $R(Q) \subset N(P)$. On the other hand if $P\mathbf{v} = 0$ Then $Q\mathbf{v} = \mathbf{v} - P\mathbf{v} = \mathbf{v}$. Note if P_1 is the projector of the proceeding section then P_2 is the complementary projector.

As we saw given a pair of complementary spaces E_1 and E_2 we obtain complementary projectors P_1 and P_2. The converse is also true, namely, suppose that we are given a projector P an $n \times n$ matrix and let Q be the complementary projector. The previous fact allows us to see immediately that $R(P)$ and $R(Q)$ are complementary subspaces of $\mathbb{C}^n$. Indeed, for any vector $\mathbf{w} \in \mathbb{C}^n$ we have $\mathbf{w} = P\mathbf{w} + Q\mathbf{w}$. To check this, we need to check that any vector $\mathbf{w}$ can be written as the sum of avector in the range of P and a vector in the range of Q and that this is unique. The first is obvious $P\mathbf{w} + Q\mathbf{w} = P\mathbf{w} + (I-P)\mathbf{w} = \mathbf{w}$. The second uses our fact.

I.e. if $\mathbf{w} \in R(P) \cap R(Q)$ then $\mathbf{w} \in R(Q)$. But then by above $\mathbf{w} \in N(P)$. But $\mathbf{w} \in R(P)$ so $\mathbf{w} = P\mathbf{v}$ so $0 = P\mathbf{w} = P^2\mathbf{v} = P\mathbf{v} = \mathbf{w}$.

That is given a projector we obtain a pair of complementary subspaces for which the projector tells us how to decompose.

5. Orthogonal Projectors Revisited

We now return to the concept of *Orthogonal Projector*. We say a projector is orthogonal provided the $R(P)$ and $R(Q)$ are orthogonal subspaces, i.e. are orthogonal complements. It is important to note that orthogonal projectors are *NOT* orthogonal or unitary matrices.

They are however hermitian matrices and in fact this characterizes them. That is we have the following

THEOREM 5.1. *A projector P is an orthogonal projector if and only if $P^* = P$.*

PROOF. ($\Leftarrow$) Let $Q = I - P$ be complementary projector. If $\mathbf{w} \in R(Q)$ then $\mathbf{w} = Q\mathbf{v} = \mathbf{v} - P\mathbf{v}$ for some $\mathbf{v}$. Now let $\mathbf{a} \in R(P)$ so $\mathbf{a} = P\mathbf{b}$. Then $\langle \mathbf{w}, \mathbf{a} \rangle = \langle \mathbf{v} - P\mathbf{v}, P\mathbf{b} \rangle = \langle P^*\mathbf{v} - P^*P\mathbf{v}, \mathbf{b} \rangle$ Now using $P = P^*$ this gives $\langle P\mathbf{v} - P^2\mathbf{v}, \mathbf{b} \rangle = \langle 0, \mathbf{b} \rangle = 0$. ($\Rightarrow$) In order to show this we must use the fact that any $E \subset \mathbb{C}^n$ a vector space admits an orthonormal basis. We will show this in a couple of lectures. We know that $R(P)$ and $R(Q)$ are orthogonal compliments. Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be an orthonormal basis of P and let $\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n$ be an orthonormal basis of $R(Q)$. Notice that then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is then an orthonormal basis of $\mathbb{C}^n$. Now, $P\mathbf{x}_j = 0$ for $k+1 \leq j \leq n$ as such $\mathbf{x}_j \in R(Q) = Null(P)$. On the other hand $P\mathbf{x}_j = \mathbf{x}_j$ for $1 \leq j \leq q$. This is because $\mathbf{x}_j = P\mathbf{x}'_j$ but then $P\mathbf{x}_j = P^2\mathbf{x}'_j = P\mathbf{x}'_j = \mathbf{x}_j$. Thus, in terms of the basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ P looks pretty nice. Let $X = \begin{bmatrix} \mathbf{x}_1 | & \cdots & | \mathbf{x}_n \end{bmatrix}$ be the $n \times n$ matrix with columns $\mathbf{x}_i$. Notice that X is unitary. Given a general vector $\mathbf{v}$, $\mathbf{v} = \sum_j c_j \mathbf{x}_j$ where $\mathbf{c} = X^{-1}\mathbf{v} = X^*\mathbf{v}$. In particular, $P\mathbf{v} = P(\sum_j c_j \mathbf{x}_j) = XId_k\mathbf{c} = XId_kX^*\mathbf{v}$. Here Id_k is matrix with 1 along diagonal for first k rows and then 0s elsewhere. In other words $P = XId_kX^*$. Then $P^* = (XId_kX^*)^* = (X^*)^*Id_k^*X^* = XId_kX^* = P$. $\square$

One final remark: Given a subspace E, there are lots of complementary subspaces. These correspond to different oblique projectors P with $R(P) = E$. However, it is not too hard to see that there is only one orthogonal projector $P^\perp$ with $R(P^\perp) = E$. Equivalently, there is only one complementary subspace that is orthogonal.

CHAPTER 2

Eleventh and Twelfth Lectures

In this lecture I started talking about the four fundamental spaces associated to a matrix.

1. Orthogonal Complement

One bit of notation I do want to introduce. Given $E \subset \mathbb{C}^n$ a vector space I will denote by

$$E^{\perp} = \{\mathbf{v} \in \mathbb{C}^n : \langle \mathbf{v}, \mathbf{w} \rangle = 0, \mathbf{w} \in E\}$$

this is the orthogonal complement of E. It is clear that $E^{\perp}$ is a vector space and that E and $E^{\perp}$ are orthogonal. We claim also that $E + E^{\perp} = \mathbb{C}^n$, that is $E, E^{\perp}$ are orthogonal complements. We also claim that if P is an orthogonal projector with $R(P) = E$ then $E^{\perp} = R(I - P)$. Similarly, given E there is exatly one orthogonal projector P so that $R(P) = E$ and then $E^{\perp} = R(I - P)$.

2. Four fundamental spaces of a matrix

Let A be a $m \times n$ matrix. We then have two natural vector spaces associated to A. Namely $N(A) \subset \mathbb{C}^n$ and $R(A) \subset \mathbb{C}^m$. The null space and column space of A. Notice that it is important to think of these as being in *different* spaces (even if $m = n$). We now introduce two more important subspaces associated to A as we will see these turn out to be orthogonal complements of the original two.

The first of these is the *row space* of A. We denote this by $Row(A) \subset \mathbb{C}^n$ and we set $Row(A) := R(A^*)$. Notice this is essentially the span of the rows, however we have taken an adjoint in order to make the rows vectors. The second of these is called the *Left Null Space* and is denoted by $L - Null(A) \subset \mathbb{C}^m$ and we set $L - Null(A) := N(A^*)$.

Notice that the row space sits in $\mathbb{C}^n$ along with $N(A)$, while the left null space sits in $\mathbb{C}^m$ along with $R(A)$. We justify the terminology for left null space as follows: Basically it consists the rows which when multipliel agains A on the left give 0 (using the adjoint to turn rows into vectors).

It turns out that $Row(A) = R(A^*)$ and $N(A)$ are orthogonal complements in $\mathbb{C}^n$ and $L - Null(A)$ and $R(A)$ are orthogonal complements in $\mathbb{C}^m$. That is $Row(A) = N(A)^{\perp}$ and $L - Null(A) = R(A)^{\perp}$. Taken together all four spaces are known as the four fundamental spaces of the matrix A.

To prove this it suffices to restrict attention to $N(A)$ and $R(A^*)$. As we will see this is enought. We first verify these two spaces are orthogonal: Take $\mathbf{v} \in R(A^*)$ and $\mathbf{w} \in N(A)$. So $\mathbf{v} = A^*\mathbf{v}'$ for $\mathbf{v}' \in \mathbb{C}^m$. Then $\langle \mathbf{v}, \mathbf{w} \rangle = \langle A^*\mathbf{v}', \mathbf{w} \rangle = \langle \mathbf{v}', A\mathbf{w} \rangle = \langle \mathbf{v}', 0 \rangle = 0$.

In order to complete the claim we must still show that $R(A^*) + N(A) = \mathbb{C}^n$. To do this we will actually show something about the dimension of these spaces.

Namely, $dimR(A^*)+dimN(A)=n$ i.e. the dimension of the row space is the same as the dimension of the null space.

To see this we use Gaussian elimination. The point is that for each *column operation* we can do on A there is a corresponding *row operation* we can do on A^*. More precisely, suppose one gets $B\in\mathbb{C}^{m\times n}$ from A by a column operation. Then one gets B^* from A^* by a row operation.

As an example consider

$$A=\begin{bmatrix}\mathbf{a}_1| & \cdots & |\mathbf{a}_n\end{bmatrix}$$

and let

$$B=\begin{bmatrix}\mathbf{a}_1+\mathbf{a}_2| & \cdots & \mathbf{a}_n\end{bmatrix}$$

(i.e adding second column to the first) then

$$A^*=\begin{bmatrix}\mathbf{a}_1^*\\ \vdots\\ \mathbf{a}_n^*\end{bmatrix}$$

and

$$B^*=\begin{bmatrix}(\mathbf{a}_1+\mathbf{a}_2)^*\\ \vdots\\ \mathbf{a}_n^*\end{bmatrix}=\begin{bmatrix}\mathbf{a}_1^*+\mathbf{a}_2^*\\ \vdots\\ \mathbf{a}_n^*\end{bmatrix}$$

which is adding second row to the first. Similarly row operations on A become row column operations on A^*.

A consequence of this fact is that $(rref(A))^*=cref(A^*)$. That is:

LEMMA 2.1. *Let $A\in\mathbb{C}^{m\times n}$ then $(rref(A))^*=cref(A^*)$ and $(cref(A))^*=rref(A^*)$.*

PROOF. It is straightforward to check that if a matrix B is in row reduced echelon form (rref) then B^* is in column reduced echelon form (cref). (Go back to the definition to convince yourself). Since hence $rref(A)^*$ is in cref and since $rref(A)$ is obtained from A by a finite number of row operations, $rref(A)^*$ is obtained from A^* by a finite number of column operations. By the uniqueness of $cref(A^*)$ we then see that $cref(A^*)=rref(A)$. ☐

As a consequence fo this, the number of pivots in $rref(A)$ is the same as the number of pivots of $cref(A*)$ and number of pivots of $cref(A)$ is same as number of pivots of $rref(A*)$. An important fact we have already used was that for an arbitrary $m\times n$ matrix B, $dimR(B)$ was the number of pivots (say k) of $cref(B)$. Similarly, if the number of pivots of $rref(B)$ is l then $dimN(A)=n-l$. Hence $dimN(A)$ is $n-k$ where k is the number of pivots of $rref(A)$. However, $rref(A)^*$ has the same number of pivots as $cref(A^*)$ and so we have $dimR(A^*)=k$. Hence $dimN(A)+dimRow(A)=n$ as claimed.

We can now show that $R(A^*)$ and $N(A)$ are orthogonal complements. Notice we've already shown they are orthogonal. Pick a basis $\mathbf{v}_1,\dots,\mathbf{v}_k$ of $R(A^*)$ and a basis $\mathbf{v}_{k+1},\dots,\mathbf{v}_n$ of $N(A)$. Notice the numbers of vectors is right as $dimRow(A)+dimN(A)=n$. We claim $\mathbf{v}_1,\dots,\mathbf{v}_n$ is a basis. Its enough to check that it is linearly independent. Suppose that $\sum_{i=1}^n c_i\mathbf{v}_i=0=\sum_{i=1}^k c_i\mathbf{v}_i+\sum_{i=k+1}^n c_i\mathbf{v}_i$. But then (by uniqueness of the decomposition) $\sum_{i=1}^k c_i\mathbf{v}_i=0$ and $\sum_{i=k+1}^n c_i\mathbf{v}_i$. Then by linear independence in $R(A^*)$ and $N(A)$ $c_i=0$ for all i.

We now can conclude that $L-Null(A)$ and $R(A)$ are orthogonal complements. To see this it is enough to notice that $R(A) = Row(A^*)$ and $L-Null(A) = N(A^*)$. And use what we already showed only for A^* instead of A.

3. Relations amongst the Fundamental Spaces

We can now get useful relationships between the sizes of the fundamental spaces of A.

THEOREM 3.1. *Let $A \in Cx^{m\times n}$ then $dimR(A) = dimR(A^*)$ i.e the dimension of the row space is the same as that of the column space.*

PROOF. Pick $\mathbf{v}_1, \ldots, \mathbf{v}_k$ a basis of $R(A^*)$ and $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ a basis of $N(A)$. As we saw above the set $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of $\mathbb{C}^n$. Now let $\mathbf{w}_i = A\mathbf{v}_i$. Notice that $\mathbf{w}_i = 0$ for $k+1 \leq i \leq n$. We claim however, that for $1 \leq i \leq k$, the $\mathbf{w}_i$ form a basis of $R(A)$. Lets check they are linearly independent. Suppose $0 = \sum_{i=1}^k c_i\mathbf{w}_i = A\sum_{i=1}^k c_i\mathbf{v}_i$. Hence $\mathbf{v} = \sum_{i=1}^k c_i\mathbf{v}_i \in N(A)$. But $\mathbf{v} \in R(A^*)$ (by our set up) so must have $\mathbf{v} = 0$. However, as the $\mathbf{v}_i$ are a basis, all the $c_i = 0$. Let's check they span $R(A)$. Pick $\mathbf{w} \in R(A)$. Write $\mathbf{w} = R\mathbf{v}$. Now as $N(A)$ and $R(A^*)$ are complementary in $\mathbb{C}^n$ we can write $\mathbf{v} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a} \in N(A)$ and $\mathbf{b} \in R(A^*)$. Then $\mathbf{w} = A\mathbf{v} = A(\mathbf{a}+\mathbf{b}) = A\mathbf{a} + A\mathbf{b} = A\mathbf{b}$. Now write $\mathbf{b} = \sum_{i=1}^k c_i\mathbf{v}_i$. Then $\mathbf{w} = A\mathbf{b} = \sum_{i=1}^k c_i\mathbf{w}_i$ so the $\mathbf{w}_i$ span $R(A)$. Thus $dimR(A) = k = dimR(A^*)$. $\square$

COROLLARY 3.2. *Let $A \in \mathbb{C}^{m\times n}$ then $dimR(A)$ is the number of pivots in $rref(A)$.*

COROLLARY 3.3. *(Rank-Nullity Theorem) Let $A \in \mathbb{C}^{m\times n}$ then $dimR(A) + dimN(A) = n$.*

4. Other Facts about the fundamental spaces

A good example of using the four fundamental subspaces is the following fact:

PROPOSITION 4.1. *Let $A \in \mathbb{C}^{m\times n}$ then $A^*A\mathbf{v} = 0$ if and only if $A\mathbf{v} = 0$.*

PROOF. If $A\mathbf{v} = 0$ then it is clear that $A^*A\mathbf{v} = 0$. On the other hand, if $A^*A\mathbf{v} = 0$ then $A\mathbf{v}$ is in $N(A^*)$ i.e. in $L - Null(A)$. On the other hand $A\mathbf{v}$ is clearly in $R(A)$. That is $A\mathbf{v} \in L - Null(A) \cap R(A)$ but these two spaces are complements so $A\mathbf{v} = 0$. $\square$

Let us pick out a matrix factorization from the proof of Theorem 3.1. Pick an orthonormal basis $\mathbf{v}_i$ of $\mathbb{C}^n$ so that $\mathbf{v}_1, \cdots, \mathbf{v}_k$ is an orthonormal basis of $R(A^*)$ and $\mathbf{v}_{k+1}, \cdots, \mathbf{v}_n$ is an orthonormal basis of $N(A)$ (well see why we can do this next lecture). Similarly, pick a basis of $\mathbb{C}^m$ $\mathbf{w}_j$ so that $\mathbf{w}_1, \ldots, \mathbf{w}_k$ is a orthonormal basis of $R(A)$ and $\mathbf{w}_{k+1}, \ldots, \mathbf{w}_m$ is an orthonoraml basis of $L - Null(A)$. Then

$$A = W\begin{bmatrix}\hat{A} & 0\\ 0 & 0\end{bmatrix}V^{-1} = W\begin{bmatrix}\hat{A} & 0\\ 0 & 0\end{bmatrix}V^*$$

Where $\hat{A}$ is a $k\times k$ non-singular matrix. And the values 0 specify a $k\times(n-k)$ matrix with all zero entries, a $(m-k)\times k$ matrix with all zero entries and a $(m-k)\times(n-k)$ matrix with all zero entries.