1. (20 points) Consider the following matrices that depend on a parameter \( \lambda \in \mathbb{C} \):

\[
A_\lambda = \begin{bmatrix}
1 & 0 \\
2 & 0 \\
\lambda & -1
\end{bmatrix}
\]

(a) For each \( \lambda \), determine \( ||A_\lambda||_F \) the Frobenius norm of \( A_\lambda \).

(Note: technically \( \lambda \in \mathbb{C} \) which complicates things). The Frobenius norm of \( A_\lambda \) is the square-root of the sum of the squares (of the modulus) of the entries of \( A_\lambda \).

\[
||A_\lambda||_F = \sqrt{1^2 + 2^2 + (-1)^2 + |\lambda|^2} = \sqrt{6 + |\lambda|^2}
\]

(b) For each \( \lambda \), determine \( ||A_\lambda||_2 \) the induced 2-norm of \( A_\lambda \).

In order to compute the two norm of \( A_\lambda \) we compute the largest singular value \( \sigma_1 \) of \( A \). This is because, by definition, \( ||A_\lambda||_2 = \sigma_1 \). We have established in class that the singular values are the square roots of the eigenvalues of the matrix

\[
B_\lambda = A_\lambda^* A_\lambda = \begin{bmatrix}
5 + |\lambda|^2 & -\lambda \\
-\lambda & 1
\end{bmatrix}
\]

The characteristic polynomial of this matrix is \( p_B(z) = det(B - zI) = z^2 - (6 + |\lambda|^2)z + 5 \) so by the quadratic formula the eigenvalues are

\[
\alpha_\pm = \frac{6 + |\lambda|^2 \pm \sqrt{|\lambda|^4 + 12|\lambda|^2 + 16}}{2}
\]

Hence, the largest singular value of \( A \) is

\[
\sigma_1 = \sqrt{\frac{6 + |\lambda|^2 + \sqrt{|\lambda|^4 + 12|\lambda|^2 + 16}}{2}}
\]
2. (20 points) Let $A \in \mathbb{C}^{m \times m}$ be hermitian (i.e. $A = A^\ast$). Let $P \in \mathbb{C}^{m \times m}$ be the matrix representing orthogonal projection onto $N(A)$. Please show that $X = A + P$ is invertible. (Hint: Think about the four fundamental subspaces of $A$).

In order to show that $X$ is invertible is suffices to show that $N(X) = \{0\}$. To that end we note that $R(P) = N(A)$ by definition of $P$ and that $R(A) = R(A^\ast)$ as $A$ is hermitian. Now suppose that $x \in N(X)$ so $Xx = 0$ then $(A + P)x = 0$ so that $y = Ax = -Px$. In other words, $y \in R(A)$ and $y \in R(P)$ and so $y \in N(A) \cap R(A^\ast)$. As $N(A)$ and $R(A^\ast)$ are complementary spaces this means that $y = 0$. Hence, $x \in N(A)$ and so $x \in R(P)$. But then $x = Px = y = 0$. 

3. (20 points) Let \( A \in \mathbb{R}^{4 \times 2} \) be the matrix

\[
A = \begin{bmatrix}
1 & -7 \\
1 & 1 \\
3 & -7 \\
5 & -9
\end{bmatrix}
\]

(a) Find a reduced QR factorization of \( A \) i.e. \( A = \hat{Q}\hat{R} \).

To begin the QR factorization algorithm we normalize the first column of \( A \) this yields

\[
q_1 = \frac{1}{6} \begin{bmatrix}
1 \\
1 \\
3 \\
5
\end{bmatrix}
\]

and \( r_{11} = 6 \) the length of the first column. We next compute the inner product between \( q_1 \) and the second column to obtain \( r_{12} = -12 \) then subtracting \( r_{12}q_1 \) from the second column yields

\[
\hat{q}_2 = \begin{bmatrix}
-5 \\
3 \\
-1 \\
1
\end{bmatrix}
\]

This has length 6 so that \( r_{22} = 6 \) and

\[
q_2 = \frac{1}{6} \begin{bmatrix}
-5 \\
3 \\
-1 \\
+1
\end{bmatrix}
\]

Hence

\[
\hat{Q} = \frac{1}{6} \begin{bmatrix}
1 & -5 \\
1 & 3 \\
3 & -1 \\
5 & 1
\end{bmatrix}
\]

and

\[
\hat{R} = \begin{bmatrix}
6 & -12 \\
0 & 6
\end{bmatrix}
\]

and the reduced QR factorization of \( A \) is \( A = \hat{Q}\hat{R} \).
(b) Solve the following overdetermined system in the sense of least squares:

\[ Ax = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \]

By the QR factorization we know that orthogonal projection onto \( R(A) \) is given by \( \hat{Q}\hat{Q}^* \). In particular, since \( A = \hat{Q}\hat{R} \) to solve \( Ax = b \) in the sense of least squares, we solve

\[ \hat{Q}\hat{R}x = \hat{Q}\hat{Q}^*b \]

that is to solve

\[ \hat{R}x = \hat{Q}^*b. \]

In our situation, the RHS is

\[ \begin{bmatrix} 2/3 \\ -1 \end{bmatrix} \]

and solving by back substitution gives

\[ x = \begin{bmatrix} -2/9 \\ -1/6 \end{bmatrix} \]
4. (30 points) Let \( A \in \mathbb{C}^{m \times m} \) be a square matrix. Order the singular values \( \sigma_i \) of \( A \) by \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0 \) and order the eigenvalues \( \lambda_i \) of \( A \) so \( |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_m| \geq 0 \).

(a) Show that \( \sigma_1 \geq |\lambda_1| \).

Let \( x \in \mathbb{C}^m \) be the eigenvector associated to \( \lambda_1 \) so that \( Ax = \lambda_1 x \). We may normalize \( x \) so that \( ||x||_2 = 1 \). We then have

\[
||Ax||_2 = ||\lambda_1 x||_2 = |\lambda_1||x||_2 = |\lambda_1|
\]

As a consequence,

\[
|\lambda_1| \leq \sup_{x \neq 0} \frac{||Ax||_2}{||x||_2} = ||A||_2 = \sigma_1
\]

(b) Show that \( \sigma_m \leq |\lambda_m| \) (Hint: Write \( x_m \), an eigenvector associated to \( \lambda_m \), in terms of the right singular vectors \( v_1, \ldots, v_m \) of \( A \)).

Let \( A = U\Sigma V^* \) be a SVD of \( A \). We let \( v_1, \ldots, v_m \) be the columns of \( V \), i.e. the right singular vectors. Now suppose that \( x_m \) is a (non-zero) vector so that \( Ax_m = \lambda_m x_m \). We may normalize so that \( ||x_m||_2 = 1 \). Writing \( x_m \) in terms of the ONB given by the \( v_i \) one has

\[
x_m = \sum_{i=1}^m c_i v_i
\]

Note that since \( ||x_m||_2 = 1 \) one has \( \sum_{i=1}^m |c_i|^2 = 1 \) by the Pythagorean theorem. As a consequence \( Ax_m = \sum_{i=1}^m \sigma_i c_i u_i \). Hence by the Pythagorean theorem

\[
|\lambda_m|^2 = ||Ax_m||_2^2 = \sum_{i=1}^m \sigma_i^2 |c_i|^2
\]

Since \( \sigma_i \geq \sigma_1 \) one obtains

\[
|\lambda_m|^2 \geq \sum_{i=1}^m \sigma_i^2 |c_i|^2 = \sigma_1^2.
\]
(c) Using part a), show that if $\|A\|_2 < 1$ then $I + A$ is nonsingular. Here $\|A\|_2$ is the induced 2-norm and

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the $3 \times 3$ identity matrix.

(Note that $I$ should really be the $m \times m$ identity otherwise the problem makes no sense.) Since $\|A\|_2 < 1$ we have that $\sigma_1 < 1$ and so by part a) we have that $|\lambda_1| < 1$. In particular, if $\lambda \in \Lambda(A)$ then $|\lambda| < 1$. Since $\lambda \in \Lambda(I + A)$ if and only if $\lambda - 1 \in \Lambda(A)$ and $1 > |\lambda - 1| \geq 1 - |\lambda|$ we see that $|\lambda| > 0$. In other words no eigenvalue of $I + A$ is zero. As a consequence $N(I + A) = \{0\}$ and so $I + A$ is invertible.
5. (20 points) Let \( A \in \mathbb{C}^{3 \times 3} \) be the matrix
\[
A = \begin{bmatrix}
-1 & 3 & -2 \\
0 & 3 & 1 \\
0 & 4 & 1
\end{bmatrix}
\]

Find a unitary matrix \( Q \in \mathbb{C}^{3 \times 3} \) so that
\[
QA = \begin{bmatrix}
1 & * & * \\
0 & * & * \\
0 & 0 & *
\end{bmatrix}
\]

Here * represents an unspecified number.

Notice that the first column of \( A \) is \(-e_1\) and of \( QA \) is \( e_1\). Hence, we must have \( Q(-e_1) = e_1\). That is the first column of \( e_1 \) is \(-e_1\). For \( Q \) to be unitary it must have orthonormal columns and hence \( Q \) has the form
\[
Q = \begin{bmatrix}
-1 & 0 \\
0 & Q'
\end{bmatrix}
\]
where \( Q' \in \mathbb{C}^{2 \times 2} \) is unitary. In addition, we want to have
\[
Q'
\begin{bmatrix}
3 \\
4
\end{bmatrix} = \begin{bmatrix}
x \\
0
\end{bmatrix}
\]

Now for \( Q' \) to be unitary it must preserve length. In particular, one must have \( |x| = \sqrt{3^2 + 4^2} = 5 \). We take \( x = 5 \).

We now have a number of choices we could make in finding \( Q' \). In the spirit of the Householder algorithm we take \( Q' \) to be a reflection. In this case we set
\[
v = \begin{bmatrix}
3 \\
4
\end{bmatrix} - \begin{bmatrix}
5 \\
0
\end{bmatrix} = \begin{bmatrix}
-2 \\
4
\end{bmatrix}
\]
be the vector normal to the line midway between \([3, 4]^{\top}\) and \([5, 0]^{\top}\). The orthonormal projection onto \( \text{span}(v) \) is given by
\[
P = \frac{vv^*}{||v||_2^2} = \frac{1}{5} \begin{bmatrix}
1 & -2 \\
-2 & 4
\end{bmatrix}
\]

Then we have (as we saw in class or can easily convince ourselves) that \( Q' \) is given by
\[
Q' = I - 2P = \frac{1}{5} \begin{bmatrix}
3 & 4 \\
4 & -3
\end{bmatrix}
\]

Which we verify has the desired properties. Hence
\[
Q = \frac{1}{5} \begin{bmatrix}
-5 & 0 & 0 \\
0 & 3 & 4 \\
0 & 4 & -3
\end{bmatrix}
\]
6. (30 points) (a) Let $T \in \mathbb{C}^{m \times m}$ be upper triangular. Show that if $T$ is unitary then $T$ is diagonal. (Hint: Use the fact that columns are orthogonal and induct on $m$).

We prove the result by induction on $m$. When $m = 1$ then any matrix is diagonal so we are done. Assume the result is true for all $m \times m$ upper-triangular matrices. We wish to prove it true for $(m + 1) \times (m + 1)$ upper-triangular matrices.

To that end we note that if $T \in \mathbb{C}^{(m+1) \times (m+1)}$ is upper triangular and unitary. Then the first column of $T$ must be of the form $\mu e_1$ (as $T$ upper triangular) and the length of the first column is 1 (as $T$ unitary) so $|\mu| = 1$. Since every other column of $T$ is orthogonal to the first column (as $T$ is unitary) $T$ has the form

$$T = \begin{bmatrix} \mu & 0 \\ 0 & T' \end{bmatrix}.$$ 

Where $T' \in \mathbb{C}^{m \times m}$ is upper triangular and unitary. In particular, by the induction hypothesis $T'$ is diagonal and hence so is $T$.

(b) Let

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{mm} \end{bmatrix} \in \mathbb{C}^{m \times m}$$

be a diagonal matrix. Show that if $A$ is unitary then $|a_{ii}| = 1$ for $1 \leq i \leq m$.

Since the columns of a unitary matrix must be of unit length it is straightforward to see that $|a_{ii}| = 1$. 

(c) Let \( X \in \mathbb{C}^{m \times m} \) be unitary, use parts a), b) and the Schur factorization to show that \( X \) is unitarily diagonalizable (i.e. \( \mathbb{C}^m \) has an orthonormal basis of eigenvectors) and that \( \lambda \in \Lambda(X) \) implies \( |\lambda| = 1 \).

Write the Schur factorization of \( X \) as

\[
X = QTQ^* 
\]

where \( Q \in \mathbb{C}^{m \times m} \) is unitary and \( T \in \mathbb{C}^{m \times m} \) is diagonal. For \( X \) to be unitary one has \( X^* = X^{-1} \). On the one hand

\[
X^* = (QTQ^*)^* = (Q^*)^*T^*Q^* = QT^*Q. 
\]

on the other

\[
X^{-1} = (QTQ^*)^{-1} = (Q^*)^{-1}T^{-1}Q^{-1} = QT^{-1}Q^*.
\]

Hence \( T \) is unitary and so by a) and b) is diagonal with entries on the diagonal all of length 1. In all cases the diagonal entries of \( T \) are the eigenvalues of \( X \) and in this case the columns of \( Q \) are the eigenvectors of \( X \) and so we have proved the claim.
7. (30 points) Let $A \in \mathbb{R}^{3 \times 3}$ be the matrix

$$A = \begin{bmatrix} -3 & 2 & -2 \\ 0 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

(a) Find the eigenvalues of $A$ and give their algebraic multiplicity.

By expanding along the middle column we see that the characteristic polynomial of $A$ is:

$$p_A(z) = \det(zI - A) = (z - 1)((z + 3)(z - 2) - 2(-2)) = (z - 1)(z^2 + z - 6 + 4).$$

By inspection (or using the quadratic formula) this can be factored as

$$p_A(z) = (z - 1)(z - 1)(z + 2).$$

Thus, the eigenvalues are $\lambda = -2$ with algebraic multiplicity 1 and $\lambda = 1$ with algebraic multiplicity 2.

(b) Verify that $A$ is diagonalizable and find a basis of eigenvectors.

As

$$A - (-2)I = \begin{bmatrix} -1 & 2 & -2 \\ 0 & 3 & 0 \\ 2 & -1 & 4 \end{bmatrix}$$

we can do Gaussian elimination to see that an eigenvector associated to $\lambda = -2$ is

$$x_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Similarly, as

$$A - I = \begin{bmatrix} -4 & 2 & -2 \\ 0 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

Gaussian elimination shows that one has (linearly independent) eigenvectors

$$x_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

one verifies that $x_1, x_2, x_3$ form a linearly independent set and since $\dim \mathbb{C}^3 = 3$ they must form a basis.
(c) Determine the matrices $X$ and $\Lambda$ so that $X$ is non-singular and $\Lambda$ is diagonal and so one has a factorization:

$$A = X\Lambda X^{-1}$$

If we let

$$X = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$ 

Using Gaussian elimination one computes

$$X^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 1 & -1 \end{bmatrix}.$$ 

Hence using the fact that the columns of $X$ are eigenvectors of $A$ one has

$$A = \frac{1}{3} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 1 & -1 \end{bmatrix}.$$ 

(d) Compute $A^n e_1$ for $n \geq 1$ an integer. Please simplify your answer as much as possible. Here

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

As $A = X\Lambda X^{-1}$ one computes that

$$A^n = X\Lambda^n X^{-1}.$$ 

That is

$$A^n = \frac{1}{3} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-2)^n \end{bmatrix} \begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 1 & -1 \end{bmatrix}.$$ 

Hence,

$$A^n e_1 = \frac{1}{3} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ (-2)^{n+1} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 + (-2)^{n+2} \\ 0 \\ 2 + (-2)^{n+1} \end{bmatrix}.$$
8. (30 points) Let \( A \in \mathbb{R}^{2 \times 2} \) be the following matrix
\[
A = \begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix}
\]
Let \( S_p = \{ x \in \mathbb{R}^2 : \|x\|_p = 1 \} \). Let \( AS_p = \{ Ax \in \mathbb{R}^2 : x \in S_p \} \). Here \( 1 \leq p \leq \infty \) and \( \| \cdot \|_p \) is the \( p \)-norm on \( \mathbb{R}^2 \).

(a) Compute \( \mu_1 = \|A\|_1 \) the induced 1-norm of \( A \) and \( \mu_\infty = \|A\|_\infty \) the induced \( \infty \)-norm of \( A \). Remember to justify your computation.

For a vector \( x \in S_1 \) let us write \( x = \alpha e_1 + (1 - \alpha) e_2 \) where we assume \( 0 \leq \alpha \leq 1 \) (so we assume \( x \) in first quadrant, by symmetry this is enough). Then \( Ax = (2 + \alpha)e_1 + (-2 + 3\alpha)e_2 \) then \( \|Ax\|_1 = |2 + \alpha| + |-2 - 3\alpha| \). When \( \alpha > 2/3 \) this is equal to \( 2 + \alpha - 2 + 3\alpha = 4\alpha \), while for \( \alpha \leq 2/3 \) this is equal to \( 2 + \alpha + 2 - 3\alpha = 4 - 2\alpha \). Notice that this is maximized for \( \alpha = 0 \) or \( \alpha = 1 \) and has maximum value \( \mu_1 = 4 \).

For a vector \( y \in S_\infty \) let us write \( y = xe_1 + ye_2 \) where \( x = 1 \) and \( 0 \leq y \leq 1 \) and \( 0 \leq x < 1 \) (so again we are in the first quadrant). Then \( Ay = (3x + 2y)e_1 + (x - 2y)e_2 \). Then \( \|Ay\|_\infty = \max \{|3x + 2y|, |x - 2y|\} \). By inspection one sees that the maximum value is \( \mu_\infty = 5 \).

(b) Determine all vectors \( x_1 \in S_1 \) and \( x_\infty \in S_\infty \) so that \( \|Ax_1\|_1 = \mu_1 \) and \( \|Ax_\infty\|_\infty = \mu_\infty \).

In the previous problem we see that \( x_1 \) can be any of the following and no other vectors:
\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\]

In the previous problem we see that \( x_\infty \) can be any of the following and no other vectors:
\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\]
(c) Sketch $S_1$ and $A S_1$ and indicate the vectors $x_1$ and $Ax_1$ on your picture.

(d) Sketch $S_\infty$ and $A S_\infty$ and indicate the vectors $x_\infty$ and $Ax_\infty$ on your picture.