Mathematic 104, Fall 2010: Assignment #1

Due: Wednesday, October 6th

Instructions: Please ensure that your answers are legible. Also make sure that all steps are shown – even for problems consisting of a numerical answer. Bonus problems cover advanced material and, while good practice, are not required and will not be graded.

Problem #1. Consider the following 4 vectors:

\[
\begin{align*}
  v_1 &= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, &
  v_2 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, &
  v_3 &= \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, &
  v_4 &= \begin{bmatrix} 10 \\ 6 \\ 4 \end{bmatrix}
\end{align*}
\]

Let \( E = \text{Span}\{v_1, v_2, v_3, v_4\} \)

a) Can the \( v_j \) form a basis for \( E \)? Please justify your answer.

b) Determine \( \text{dim}(E) \).

c) Write down a matrix whose null space is \( E \).

d) Find a vector \( w \) so that \( \{v_1, v_2, w\} \) form a basis of \( \mathbb{C}^3 \).

Problem #2. Let \( A \) and \( B \) be 2 × 2 matrices.

a) Find \( A \) and \( B \) so that \( AB \neq BA \).

b) Now fix \( A \), and suppose that we know that \( AB = BA \) for every 2 × 2 matrix \( B \). Show that \( A \) must be a multiple of the identity matrix i.e. of the form

\[
A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI.
\]

Problem #3. Let \( v_1, \ldots, v_k \) be vectors in \( \mathbb{C}^n \). If \( w \in \mathbb{C}^n \) is not in \( \text{Span}\{v_1, \ldots, v_k\} \) what condition on the \( v_i \) ensures that \( \{w, v_1, \ldots, v_k\} \) are linearly independent? Justify your answer.

Problem #4. A matrix is said to be lower triangular if \( a_{ij} = 0 \) for all \( i < j \). Consider a generic 3 × 3 lower triangular matrix:

\[
A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\]

a) If all the terms are non-zero show that the only solution to \( Av = 0 \) is \( v = 0 \).

b) If \( a_{22} = 0 \) give a non-zero vector in \( \text{Null}(A) \) and a vector not in \( \text{Range}(A) \).

Bonus Problem. Recall that in class we saw that we could associate to any complex number \( z \in \mathbb{C} \) a vector \( v \in \mathbb{R}^2 \) as follows: write \( z = a + Ib \) for \( a, b \in \mathbb{R} \) and let \( v = ae_1 + be_2 = \begin{bmatrix} a \\ b \end{bmatrix} \). In a similar manner, to any vector \( Z \in \mathbb{C}^2 \) we can associate a vector in \( V \in \mathbb{R}^4 \):

write \( Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_1 + Ib_1 \\ a_2 + Ib_2 \end{bmatrix} \) and set \( V = \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} \).

Now suppose that \( Z' = CZ \) where

\[
C = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} c_{11} + Id_{11} & c_{12} + Id_{12} \\ c_{21} + Id_{21} & c_{22} + Id_{22} \end{bmatrix}
\]

is a 2 × 2 matrix with complex entries.

a) If \( V' \) is the vector in \( \mathbb{R}^4 \) associated to \( Z' \), find a 4 × 4 matrix, \( T \), with real entries so \( V' = TV \).

b) Prove that \( T \) is non-singular if and only if \( C \) is non-singular.