Mathematic 104, Fall 2010: Assignment #1

Due: **Wednesday, October 6th**

Instructions: Please ensure that your answers are legible. Also make sure that all steps are shown – even for problems consisting of a numerical answer. Bonus problems cover advanced material and, while good practice, are *not* required and will *not* be graded.

Problem #1. Consider the following 4 vectors:

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, v_4 = \begin{bmatrix} 10 \\ 6 \\ 4 \end{bmatrix}$$

Let $E = Span\{v_1, v_2, v_3, v_4\}$

a) Can the v_j form a basis for E? Please justify your answer.
b) Determine $dim(E)$.
c) Write down a matrix whose null space is E.
d) Find a vector w so that $\{v_1, v_2, w\}$ form a basis of $\mathbb{C}^3$.

Problem #2. Let A and B be 2×2 matrices.

a) Find A and B so that $AB \neq BA$.
b) Now fix A, and suppose that we know that $AB = BA$ for every 2×2 matrix B. Show that A must be a multiple of the identity matrix i.e. of the form

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI.$$

Problem #3. Let $v_1, \dots v_k$ be vectors in $\mathbb{C}^n$. If $w \in \mathbb{C}^n$ is not in $Span\{v_1, \dots, v_k\}$ what condition on the v_i ensures that $\{w, v_1, \dots, v_k\}$ are linearly independent? Justify your answer.

Problem #4. A matrix is said to be lower triangular if $a_{ij} = 0$ for all $i < j$. Consider a generic 3×3 lower triangular matrix:

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a) If all the terms are non-zero show that the only solution to $Av = 0$ is $v = 0$.
b) If $a_{22} = 0$ give a non-zero vector in $Null(A)$ and a vector *not* in $Range(A)$.

Bonus Problem. Recall that in class we saw that we could associate to any complex number $z \in \mathbb{C}$ a vector $v \in \mathbb{R}^2$ as follows: write $z = a + Ib$ for $a, b \in \mathbb{R}$ and let $v = ae_1 + be_2 = \begin{bmatrix} a \\ b \end{bmatrix}$. In a similar manner, to any vector $Z \in \mathbb{C}^2$ we can associate a vector in $V \in \mathbb{R}^4$:

$$\text{write } Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_1 + Ib_1 \\ a_2 + Ib_2 \end{bmatrix} \text{ and set } V = \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix}.$$

Now suppose that $Z' = CZ$ where

$$C = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} c_{11} + Id_{11} & c_{12} + Id_{12} \\ c_{21} + Id_{21} & c_{22} + Id_{22} \end{bmatrix}$$

is a 2×2 matrix with complex entries.

a) If V' is the vector in $\mathbb{R}^4$ associated to Z', find a 4×4 matrix, T, with real entries so $V' = TV$.
b) Prove that T is non-singular if and only if C is non-singular.