# Solution Manual

## Problem #1 :

a) $2\begin{bmatrix}5\\2\\3\end{bmatrix} - \begin{bmatrix}10\\6\\4\end{bmatrix} + 2\begin{bmatrix}1\\1\\0\end{bmatrix} - 2\begin{bmatrix}2\\1\\1\end{bmatrix} = 2v_3 - v_4 + 2v_2 - 2v_1 = 0$

non-trivial linear combination vanishes so they cannot be basis.

b) Systematically we have to ~~for~~ find dim (E) by Gaussian Elimination.

$$\begin{array}{cccc} \begin{bmatrix}2 & 1 & 5 & 10\\ 1 & 1 & 2 & 6\\ 1 & 0 & 3 & 4\end{bmatrix} \\ v_1 \quad v_2 \quad v_3 \quad v_4 \end{array} \xrightarrow[\substack{v_4 - 10v_2 \\ v_1 - 2v_2}]{v_3 - 5v_2} \begin{array}{c} \begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 1 & -3 & -4\\ 1 & 0 & 3 & 4\end{bmatrix} \\ c_1 \quad c_2 \quad c_3 \quad c_4 \end{array} \xrightarrow[c_2 \to c_1]{c_1 \to c_2} \begin{array}{c} \begin{bmatrix}1 & 0 & 0 & 0\\ 1 & -1 & -3 & -4\\ 0 & 1 & 3 & 4\end{bmatrix} \\ c_1 \quad c_2 \quad c_3 \quad c_4 \end{array}$$

$$\xrightarrow[\substack{c_4 + 4c_2 \\ c_1 + c_2}]{c_3 + 3c_2} \begin{array}{c} \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} \\ c_1 \quad c_2 \quad c_3 \quad c_4 \end{array} \Longrightarrow$$

So the dim of span of column is 2, dim (E) = 2

c) from computation above $E$ is generated by two vectors $e_1 = (1,0,0)$ and $e_2 = (0,-1,1)$. in order to find linear Transformation $A$, such that $A(E) = 0$, it's sufficient to have $Ae_1 = 0$ and $Ae_2 = 0$

$$A = \left[ c_1 \,\middle|\, c_2 \,\middle|\, c_3 \right], \quad \left[ c_1 \,\middle|\, c_2 \,\middle|\, c_3 \right]\begin{bmatrix}1\\0\\0\end{bmatrix} = \left[c_1\right] = 0$$

$$\left[ c_1 \,\middle|\, c_2 \,\middle|\, c_3 \right]\begin{bmatrix}0\\-1\\1\end{bmatrix} = \left[c_2\right] - \left(c_3\right] = 0 \implies \left[c_2\right] = \left[c_3\right]$$

So for example $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ works.

d) from The process of Gaussian Elimination in b) we implicitly show that Span $(v_1, v_2) =$ Span $(e_1, e_2)$ so we have to find $w$ such that to be linear independent of span $(e_1, e_2)$

$$\text{Span}(e_1, e_2) = \begin{pmatrix} t \\ -k \\ k \end{pmatrix}, \; t, k \in \mathbb{C}$$ so obviously $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is not in this space so is independent.

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## Problem 2:

(a) Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \neq BA = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

(b) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$$AB_1 = \begin{pmatrix} a_{11} & a_{11}+a_{12} \\ a_{21} & a_{21}+a_{22} \end{pmatrix} = B_1A = \begin{pmatrix} a_{11}+a_{21} & a_{12}+a_{22} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\Longrightarrow \quad a_{11} = a_{11} + a_{21} \quad \Longrightarrow \quad a_{21} = 0$$

$$a_{11} + a_{12} = a_{12} + a_{22} \Longrightarrow \quad a_{11} = a_{22}$$

$$a_{21} + a_{22} = a_{22} \quad \Longrightarrow \quad a_{21} = 0$$

$$AB_2 = \begin{pmatrix} a_{11}+a_{12} & a_{12} \\ a_{21}+a_{22} & a_{22} \end{pmatrix} = B_2A = \begin{pmatrix} a_{11} & a_{12} \\ a_{11}+a_{21} & a_{12}+a_{22} \end{pmatrix}$$

$$a_{11} + a_{12} = a_{11} \quad \Longrightarrow \quad a_{12} = 0$$

so $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

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**Problem 3**: obviously if $\{w, v_1, -, v_k\}$ are linearly independent then $v_1, -, v_k$ have to be linearly independent. But that's sufficient. Since if $\{w, v_1, -, v_k\}$ are not linearly independent

assume $\sum_{i=1}^{k} c_i v_i + c_0 w = 0$ is non-trivial linear combination

if $c_0 = 0$ then $\sum_{1}^{k} c_i v_i = 0$ has to non-trivial linear combination which contradicts the hypothesis. if $c_0 \neq 0$ then we can write

$$w = \sum_{1}^{k} \frac{c_i}{c_0} v_i \Longrightarrow w \in \operatorname{span}(v_1, \dots, v_k) \quad \text{which is not the case}$$

So $\{w, v_1, \dots, v_k\}$ are linearly ind.

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## Problem 4 :

a) the point is using Gaussian Elimination

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[R_3 - \frac{a_{31}}{a_{11}} R_1]{R_2 - \frac{a_{21}}{a_{11}} R_1} \begin{bmatrix} a_{11} & & \\ 0 & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \xrightarrow{R_3 - \frac{a_{32}}{a_{22}} R_2} \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

So if $Av = A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Longrightarrow \begin{array}{l} a_{11} v_1 = 0 \\ a_{22} v_2 = 0 \\ a_{33} v_3 = 0 \end{array} \Longrightarrow v_i = 0$ (because $a_{ii} \neq 0$)

b) $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$ let $v = \begin{bmatrix} 0 \\ -a_{33} \\ a_{32} \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[\text{Gauss Eli}]{\text{easily by}} \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix}$$

(we have assumed that the other entries are not zero)

because $\operatorname{range}(A) = \operatorname{span} \left\langle \begin{bmatrix} a_{11} \\ a_{21} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ a_{32} \end{bmatrix} \right\rangle = \begin{bmatrix} t a_{11} \\ t a_{21} \\ s a_{32} \end{bmatrix}$ $s, t \in \mathbb{C}$

So take $\begin{bmatrix}1\\0\\0\end{bmatrix} \notin$ range(A) because if it was in Range (A)

then $ta_{11} = 1$ so $t \neq 0$ but $ta_{21} = 0$ so $t = 0$ is contradiction

So $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not in Range (A)

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## Bonus Problem :

So assume $v \in \mathbb{C}^2$ Then we write $v = v_1 + i v_2$ where $v_1, v_2 \in \mathbb{R}^2$

and then $[v]_{\mathbb{R}^4} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and also assume $A \in M_2(\mathbb{C})$

then write $A = B + iC$

$$Av = (B+iC)(v_1 + iv_2) = (Bv_1 - Cv_2) + (Cv_1 + Bv_2)i$$

So $[Av]_{\mathbb{R}^4} = \begin{bmatrix} Bv_1 - Cv_2 \\ Cv_1 + Bv_2 \end{bmatrix}$

We want to find $4 \times 4$ matrix such that

$$\begin{bmatrix} [A]_{\mathbb{R}^4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Bv_1 - Cv_2 \\ Cv_1 + Bv_2 \end{bmatrix}$$

So $[A]_{\mathbb{R}^4} = \begin{bmatrix} B & -C \\ C & B \end{bmatrix}$

$$\det \begin{array}{c} \left[\begin{array}{cc} B & -C \\ \hline C & B \end{array}\right] \begin{array}{c} R_1 \\ R_2 \end{array} \\ \uparrow \\ R_1 + iR_2 \end{array} = \det \left[\begin{array}{c:c} B+iC & -C+iB \\ C & B \end{array}\right]_{\begin{array}{cc} C_1 & C_2 \end{array}} \underset{C_2+(-iC_1)}{=} \det \begin{bmatrix} B+iC & 0 \\ C & B-iC \end{bmatrix}$$

$$= \det |(B+iC)|^2 \quad \text{so} \quad |B+iC| \neq 0 \quad \text{iff} \quad \begin{bmatrix} B & -C \\ C & B \end{bmatrix} \text{ has}$$

non zero determinant.

Another way to show this without using det is using

$A$ is invertible iff $Av=0$ has trivial solution

$$\left(\text{if } Av=0 \text{ then } v=0\right) \iff \left(\text{if } Bv_1 - Cv_2 = 0 \text{ and } Cv_1 + Bv_2 = 0 \text{ then } v_1 = v_2 = 0\right)$$

$$\iff \left(\text{if } \begin{bmatrix} B & -C \\ C & B \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \text{ then } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0\right) \iff \left(\begin{bmatrix} B & -C \\ C & B \end{bmatrix} \text{ is invertible}\right)$$