Problem #1 R is uppertriangular. $\exists B$ such $RB = I$. assume that

$R = [a_{ij}]$ and $B = (b_{ij}]$ so $[RB]_{ij} = [I]_{ij} \Rightarrow \sum_{k=1}^{m} a_{ik} b_{kj} = \delta_{ij}$ (*)

note that $a_{ij} = 0$ for $i > j$

let $i = m$ in * : $a_{mm} b_{mj} = \delta_{mj} \Longrightarrow$ if $m = j$ we get $a_{mm} b_{mm} = 1$

So $b_{mm} = a_{mm}^{-1}$ if $j < m$ then $a_{mm} b_{mj} = 0$ (since $a_{mm} \neq 0$) $b_{mj} = 0$ (1)

now take $i = m-1$ in * : $a_{m-1,m-1} \cdot b_{m-1,j} + a_{m-1,m} b_{mj} = \delta_{m-1,j}$ (2)

let $j < m-1$ Then we know from (1) that $b_{mj} = 0$ so

$a_{m-1,m-1} b_{m-1,j} = 0$ but $a_{m-1,m-1} \neq 0$ (why? {hint: see what happens when $j = m-1$ in (2)}) so $b_{m-1,j} = 0$ for $j < m-1$

now by backward induction assume $b_{kj} = 0$ for $j < k$ we should prove $b_{k-1,j} = 0$ for $j < k-1$. Substitue $i = k-1$ in * :

$$a_{k-1,k-1} b_{k-1,j} + a_{k-1,k} b_{kj} + \text{—} + a_{k-1,m} b_{mj} = \delta_{k-1,j}$$

if you take $j = k-1$ we get $a_{k-1,k-1} b_{k-1,k-1} = 1$ so $a_{k-1,k-1} \neq 0$

if you take $j < k-1$ by induction hypothesis we know $b_{kj} = 0$

So $a_{k-1,k-1} b_{k-1,j} = 0 \Longrightarrow b_{k-1,j} = 0$ for $j < k-1$

So we are done

Problem 2

a) $v_1^* v_2 = \overline{(3i)}.4 + \overline{(0)}.0 + \overline{4}.3i = -12i + 0 + 12i = 0$

$v_1^* v_3 = \overline{(3i)}.0 + \overline{0}.1 + \overline{4}.0 = 0$

$v_2^* v_3 = \overline{4}.0 + \overline{0}.1 + \overline{3i}.0 = 0$

So by hermitian inner product on $\mathbb{C}^3$, v_1, v_2, v_3 are orthogonal.

$\| v_2 \| = \sqrt{v_2^* v_2} = \sqrt{\overline{4}.4 + \overline{0}.0 + \overline{3i}.3i} = \sqrt{16 + \underbrace{(-3i)(3i)}_{9}} = 5$

So v_2 and likewise v_1 are not unit vector

(b) $v_1 = 3i\, e_1 + 4 e_3$, $v_2 = 4e_1 + 3i\, e_3$, $v_3 = e_2$

$$Xv_1 = v_2 + v_3 \Longrightarrow 3iXe_1 + 4Xe_3 = 4e_1 + e_2 + 3ie_3 \xrightarrow{\div 3i} Xe_1 + \frac{4}{3i} Xe_3 = \frac{4}{3i}e_1 + \frac{1}{3i}e_2 + e_3$$

$$Xv_2 = -v_2 \Longrightarrow 4Xe_1 + 3iXe_3 = -4e_1 + (-3ie_3) \xrightarrow{\div 4} Xe_1 + \frac{3i}{4}Xe_3 = -e_1 - \frac{3i}{4}e_3$$

$$Xv_3 = v_1 + v_2 + v_3 \Longrightarrow Xe_2 = (4+3i)e_1 + (4+3i)e_3 + e_2$$

$$\Longrightarrow \left(\frac{4}{3i} - \frac{3i}{4}\right) Xe_3 = \left(\frac{4}{3i} + 1\right)e_1 + \frac{1}{3i}e_2 + \left(1 + \frac{3i}{4}\right)e_3 \Longrightarrow Xe_3 = \frac{16+12i}{25}e_1 + \frac{4}{25}e_2 + \frac{12i-9}{25}e_3$$

$$Xe_1 = \frac{12i+16}{-25} e_1 + \frac{3i}{25} e_2 + \frac{9-\cancel{25}12i}{25} e_3$$

So X with respect of e_1, e_2, e_3 is of this form

$$X = \begin{array}{ccc} \left[\begin{array}{ccc} \frac{-12i-16}{25} & 4+3i & \frac{16+12i}{25} \\ \frac{-3i}{25} & 1 & \frac{4}{25} \\ \frac{9-12i}{25} & 4+3i & \frac{12i-9}{25} \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ Xe_1 & Xe_2 & Xe_3 \end{array} \end{array}$$

<u>Problem 3</u>, if v_1, v_2, v_3 are not basis, neither are w_1, w_2, w_3

So assume v_i's form basis for $\mathbb{C}^3$ if $\{w_i's\}$ is not basis then

the matrix $[w_1|w_2|w_3]$ has rank 2 or lower.

$$[w_1|w_2|w_3] = [v_1|v_2|v_3]\begin{bmatrix} 1 & 1 & \lambda \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

by elementary elimination we calculate rank of $\begin{bmatrix} 1 & 1 & \lambda \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & \lambda \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ c_1 & c_2 & c_3 \end{array} \end{array} \xrightarrow[c_3 - \lambda c_1]{c_2 - c_1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1-\lambda \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{c_3 + (1-\lambda)c_2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & \lambda \end{bmatrix}$$

So if $\lambda \neq 0$

this is lower triangular with nonzero diagonal so by

Problem Set 1 We know it would invertible or of full rank.

So λ should be zero

if $\lambda \neq 0$ and v_1, v_2, v_3 form a basis for $\mathbb{C}^3$ Then

w_1, w_2, w_3 are linearly-independent so form a basis.

Problem 4

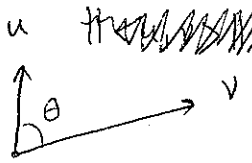
a) $\|u\|_2 = 5$, $\|v\|_2 = 13$

We know that Generally $\|a+b\|_2 \leq \|a\|_2 + \|b\|_2$

and equality occurs when a ~~…~~ and b are linearly independent.

So $\|2u+v\|_2 \leq \|2u\|_2 + \|v\|_2 = 2\|u\|_2 + \|v\|_2 = 23$

and equality occurs if u and v have same direction, in other words



$|\langle u,v\rangle| = |\, \|u\|_2 \cdot \|v\|_2 \cos(\theta) \,|$ if $\theta = 0$ Then u, v have same

direction.

$\|-2u\|_2 = \|2u\|_2$

$\|v\|_2 - \|2u\|_2 \leqslant \|2u+v\|_2$ or $\underbrace{\|v\|_2 \leqslant \|2u+v\|_2 + \|-2u\|_2}_{\text{triangle ineq.}}$

So $3 = 13 - 10 \leq \|2u+v\|_2$ and equality occurs if v and $-u$

have same direction.

b) $u^*v = \|u\|_2 \|v\|_2 \cos\theta \qquad -1 \le \cos\theta \le 1$ so

$$-65 = -\|u\|_2 \|v\|_2 \le u^*v \le \|u\|_2 \|v\|_2 = 65$$

Problem 5

(a) So the inner product of column should be zero

so (1) $\langle (\frac{3}{5}, 0, 0, \frac{4}{5}), (q_1, q_2, q_3, q_4) \rangle = \frac{3}{5} q_1 + \frac{4}{5} q_4 = 0 \Longrightarrow q_1 = \frac{-4}{3} q_4$

(2) $\langle (0, \frac{\sqrt{2}}{2}, \frac{+\sqrt{2}}{2}, 0), (q_1, q_2, q_3, q_4) \rangle = \frac{\sqrt{2}}{2} q_2 + \frac{\sqrt{2}}{2} q_3 = 0$

(3) $\langle (0, \frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, 0), (q_1, q_2, q_3, q_4) \rangle = \frac{\sqrt{2}}{2} q_2 - \frac{\sqrt{2}}{2} q_3 = 0$

$$\left.\begin{array}{l}(2)\\(3)\end{array}\right\} \Longrightarrow q_2 = q_3 = 0$$

So $(q_1, q_2, q_3, q_4) = (\frac{-4}{3} a, 0, 0, a)$

but also $\langle (q_1, q_2, q_3, q_4), (q_1, q_2, q_3, q_4) \rangle = 1$ so in case of real vector spaces we have $(\frac{-4}{3} a)^2 + a^2 = 1 \Longrightarrow a^2 = \frac{9}{25}$ so $a = \pm \frac{3}{5}$

$$(q_1, q_2, q_3, q_4) = \begin{array}{c} (-\frac{4}{5}, 0, 0, \frac{3}{5}) \\ \text{or} \\ (\frac{4}{5}, 0, 0, -\frac{3}{5}) \end{array}$$

b) again in case of Complex vector space assume

$$Q = \left[c_1 \mid c_2 \mid c_3 \mid c_4 \right] \qquad Q^* = \begin{bmatrix} \overline{c_1} \\ \hline \overline{c_2} \\ \hline \overline{c_3} \\ \hline \overline{c_4} \end{bmatrix}$$

So $Q^*Q = I$ means that

$\langle c_1, c_1\rangle = \bar{c}_1 \cdot c_1 = 1$, $\langle c_2, c_2\rangle = \bar{c}_2 \cdot c_2 = 1$, $\langle c_3, c_3\rangle = \bar{c}_3 \cdot c_3 = 1$, $\langle c_4, c_4\rangle = \bar{c}_4 \cdot c_4 = 1$

and $\langle c_i, c_j\rangle = \bar{c}_i \cdot c_j = 0$ for $i \neq j$

but since c_1, c_2, c_3 are real vectors so their inner product $\langle c_i , c_4 \rangle = \langle c_i, (q_1, q_2, q_3, q_4)\rangle = \bar{c}_i \cdot (q_1, -, q_4) =$

$c_i \cdot (q_1, q_2, q_3 \cdot q_4)$ for $i = 1, 2, 3$

So like part (a) the equations (1), (2), (3) holds.

so $(q_1, q_2, q_3, q_4) = (-\frac{4}{3}a, 0, 0, a)$

and also $\langle c_4, c_4 \rangle = 1$ so $(\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_4) \cdot (q_1, q_2, q_3, q_4) = 1$

so $(\frac{-4}{3}\bar{a})(\frac{-4}{3}a) + \bar{a} \cdot a = 1 \Longrightarrow |a|^2 = \frac{9}{25}$ so $|a| = \frac{3}{5}$

($|a|$ is positive) So any $(\frac{-4}{3}a, 0, 0, a)$ where norm of a as complex numbers. So assume $a = \frac{3}{5}e^{i\theta}$ $0 \leq \theta \leq 2\pi$

$(q_1, q_2, q_3, q_4) = (\frac{-4}{3} \cdot \frac{3}{5}e^{i\theta}, 0, 0, \frac{3}{5}e^{i\theta}) = (\frac{-4}{5}e^{i\theta}, 0, 0, \frac{3}{5}e^{i\theta})$

Bonus Problem

a), b), c) are just entry checking and super easy.

d) $$[(AB)^*]_{ij} = \overline{[(AB)]_{ji}} = \overline{\sum_k a_{jk} b_{ki}} = \sum_k \overline{a_{jk}} \cdot \overline{b_{ki}}$$

$$[B^*A^*]_{ij} = \sum_k [B^*]_{ik}[A^*]_{kj} = \sum_k \overline{b_{ki}} \cdot \overline{a_{jk}} = \sum \overline{a_{jk}} \cdot \overline{b_{ki}}$$

So $[(AB)^*]_{ij} = [B^*A^*]_{ij} \Longrightarrow (AB)^* = B^*A^*$

e) $$\langle Av, w\rangle \overset{?}{=} \langle v, A^*w\rangle$$

$A = [a_{ij}]$ $v = (v_1, -, v_n)$, $w = (w_1, -, w_m)$

$$\langle Av, w\rangle = \sum_i \overline{(Av)_i} \cdot w_i = \sum_{i,j} \overline{a_{ij} v_j} \cdot w_i$$

$$\langle v, A^*w\rangle = \sum_i \overline{v_i} \cdot (A^*w)_i = \sum_i \overline{v_i} \cdot \Big(\sum_j \overset{\overline{a_{ji}}}{\overset{\shortparallel}{[A^*]_{ij}}} w_j\Big) = \sum_{i,j} \overline{v_i}\,\overline{a_{ji}} \cdot w_j$$

$$\sum_{i,j} \overline{a_{ij}} \cdot \overline{v_j}\; w_i = \sum_{i,j} \overline{a_{ji}}\,\overline{v_i} \cdot w_j$$ (change the role of i,j)

So $\langle Av, w\rangle = \langle v, A^*w\rangle$