Problem #1. Exercise 1.3 of Lecture 1 of Trefethen-Bau.

Problem #2. Consider the following three vectors in $\mathbb{C}^3$:

$$v_1 = \begin{bmatrix} 3i \\ 0 \\ 4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 0 \\ 3i \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

a) Show that \{v_1, v_2, v_3\} is an orthogonal set. Is this set orthonormal?

b) Let $X \in \mathbb{C}^{3 \times 3}$ be a matrix so that $Xv_1 = v_2 + v_3$, $Xv_2 = -v_2$ and $Xv_3 = v_1 + v_2 + v_3$. Determine $X$. (Hint: Look for a natural orthonormal basis).

Problem #3. Let $v_1, v_2$ and $v_3$ be vectors in $\mathbb{C}^3$. Determine a value $\lambda_0 \in \mathbb{C}$ so that when $\lambda = \lambda_0$ the vectors $w_1 = v_1 + v_2$, $w_2 = v_1 - v_3$ and $w_3 = \lambda v_1 + v_2 + v_3$ are never a basis of $\mathbb{C}^3$. When $\lambda \neq \lambda_0$ what condition on the $\{v_i\}$ is necessary and sufficient so that the $\{w_i\}$ form a basis?

Problem #4. Let $u$ and $v$ be two vectors in $\mathbb{R}^3$ so that $\|u\|_2 = 5$ and $\|v\|_2 = 13$.

a) What are the largest and smallest values of $\|2u + v\|_2$?

b) What are the largest and smallest values of $\langle u, v \rangle = u^*v$?

Problem #5. Let $Q$ be the following $4 \times 4$ matrix:

$$Q = \begin{bmatrix} 3 & 0 & 0 & g_1 \\ 0 & \sqrt{2} & \sqrt{2} & q_2 \\ 0 & -\sqrt{2} & \sqrt{2} & q_3 \\ 4 & 0 & 0 & q_4 \end{bmatrix} \quad \text{and denote by } q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \quad \text{the fourth column of } Q.$$

a) Assume $Q \in \mathbb{R}^{4 \times 4}$ determine all $q \in \mathbb{R}^4$ so that $Q$ is orthogonal – that is $Q^{-1} = Q^T$.

b) Assume instead that $Q \in \mathbb{C}^{4 \times 4}$ determine all $q \in \mathbb{C}^4$ so that $Q$ is unitary – that is $Q^{-1} = Q^*$.

Bonus Problem. Verify some of the properties of the adjoint stated in class. Recall, we defined the adjoint by letting

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{C}^{m \times n} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \in \mathbb{C}^{n \times m}$$

and saying $B$ is the adjoint of $A$ when and only when $b_{ij} = \bar{a}_{ji}$ and then writing $A^* = B$.

a) For $A \in \mathbb{C}^{m \times n}$ let $a_i \in \mathbb{C}^m$ be the columns of $A$, i.e. $A = [a_1 | \cdots | a_n]$ verify $A^* = \begin{bmatrix} a_1^* \\ \vdots \\ a_n^* \end{bmatrix}$.

b) For $B \in \mathbb{C}^{m \times n}$ let $b_i \in \mathbb{C}^1 \times n$ be the rows of $B$ that is $B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ verify that $B^* = [b_1^* | \cdots | b_m^*]$.

c) Check that $(A + B)^* = A^* + B^*$, $(\lambda A)^* = \bar{\lambda} A^*$ and $(A^*)^* = A$ for $A, B \in \mathbb{C}^{m \times n}$ and $\lambda \in \mathbb{C}$. (Hint: Consider the rules for matrix addition and scalar multiplication as they apply to each entry).

d) Show that $(AB)^* = B^* A^*$ for $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times k}$. (Hint: Express the product in terms of columns and rows).

e) Show that $\langle Av, w \rangle = \langle v, A^* w \rangle$ for $v \in \mathbb{C}^n$, $w \in \mathbb{C}^m$ and $A \in \mathbb{C}^{m \times n}$.