problem #1

a) We try to find QR factorization A by Gram-Schmidt

Orthognalization.

$$A = \begin{bmatrix} 3 & 1 \\ -4 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\quad a_1 \quad a_2$$

normalize a_1, a_2 :

$$c_1 := \frac{1}{5}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} \quad c_2 := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$q_1 = c_1 \qquad q_2 = \frac{c_2 - \langle c_1, c_2 \rangle c_1}{\| c_2 - \langle c_1, c_2 \rangle c_1 \|}$$

$$q_1 = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} \qquad q_2 = \frac{5}{\sqrt{41}}\begin{bmatrix} 16/25 \\ 12/25 \\ -1 \end{bmatrix}$$

$$a_1 = 5 q_1, \quad a_2 = \frac{\sqrt{41}}{5} q_2 + \frac{3}{5} q_1$$

So
$$A = \underbrace{\begin{bmatrix} 3/5 & \frac{16}{5\sqrt{41}} \\ -4/5 & \frac{12}{5\sqrt{41}} \\ 0 & -\frac{5}{\sqrt{41}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 5 & \frac{\sqrt{41}}{5} \\ 0 & \frac{3}{5} \end{bmatrix}}_{R}$$

(b) from Thm 11.1 we have to solve:

$$A^*Ax = A^*b \quad \text{which is}$$

for $b_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$$\begin{bmatrix}3 & -4 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}3 & 1\\-4 & 0\\0 & -1\end{bmatrix}x_1 = \begin{bmatrix}3 & -4 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}25 & 3\\3 & 2\end{bmatrix}x_1 = \begin{bmatrix}3\\1\end{bmatrix}$$

$$x_1 = \frac{1}{41}\begin{bmatrix}2 & -3\\-3 & 25\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \frac{1}{41}\begin{bmatrix}3\\16\end{bmatrix}$$

for $b_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$$\begin{bmatrix}25 & 3\\3 & 2\end{bmatrix}x_2 = \begin{bmatrix}3 & -4 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-4\\0\end{bmatrix}$$

$$x_2 = \frac{1}{41}\begin{bmatrix}2 & -3\\-3 & 25\end{bmatrix}\begin{bmatrix}-4\\0\end{bmatrix} = \frac{1}{41}\begin{bmatrix}-8\\12\end{bmatrix}$$

now

$$\|Ax_1 - b_1\|_2 = \left\|\frac{1}{41}\begin{bmatrix}-16\\-12\\-16\end{bmatrix}\right\|_2$$

$$\|Ax_2 - b_2\|_2 = \left\|\frac{1}{41}\begin{bmatrix}-12\\-9\\-12\end{bmatrix}\right\|_2$$

since $\left\|\begin{bmatrix}-16\\-12\\-16\end{bmatrix}\right\|_2 > \left\|\begin{bmatrix}-12\\-9\\-12\end{bmatrix}\right\|_2$

So $\|Ax_1 - b_1\|_2 > \|Ax_2 - b_2\|_2$ so x_2 is nearer to true solution.

problem #2 : We want to find line $y=ax+b$ that is best solution in a sense that these three points lie on this line

So:

$$y = a_1 x + b_1$$

$$\begin{aligned} 1 &= 2a_1 + b_1 \\ -1 &= 0a_1 + b_1 \\ 2 &= 5a_1 + b_1 \end{aligned} \Longrightarrow \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$A \quad \vec{x} = \vec{b}$$

So like problem #1 in order to find best solution to $A\vec{x}=\vec{b}$ we have to minimum the error $r = b - Ax$ and by Thm 11.1 we know best solution satisfies: $A^*A\,x = A^*b$

$$\begin{bmatrix} 2 & 0 & 5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 5 & 1 \end{bmatrix} x = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 29 & 7 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \frac{1}{38} \begin{bmatrix} 3 & -7 \\ -7 & 29 \end{bmatrix} \begin{bmatrix} 12 \\ 2 \end{bmatrix} = \frac{1}{38} \begin{bmatrix} 22 \\ -26 \end{bmatrix}$$

Problem #2 : So our Goal is to find best solution to

(assume your line is y=ax+b)

$a+b=2$

$a(0)-$

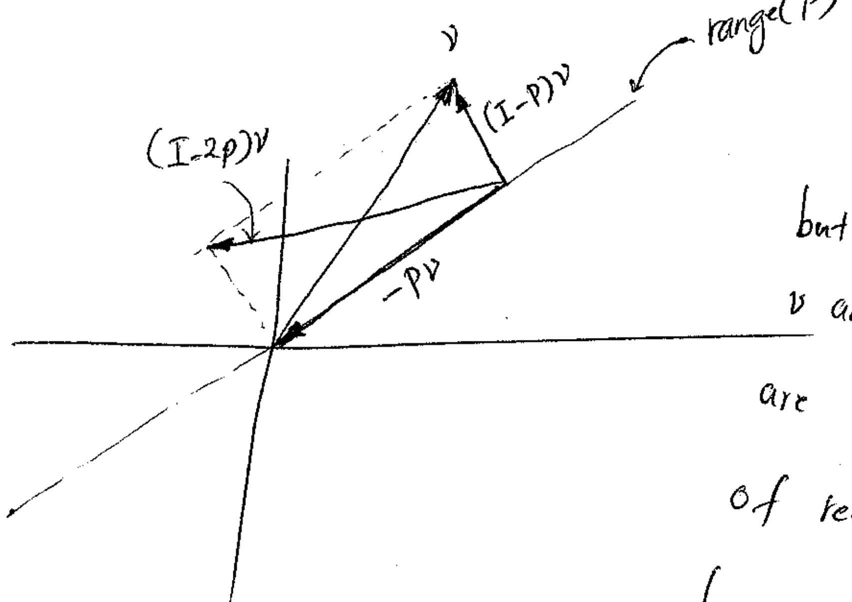
Problem # 3 : by Thm 6.1 we know $P = P^*$ so

$$(I-2P)(I-2P)^* = (I-2P)(I-2P)$$

$$= I - 4P + \underbrace{4P^2}_{P^2=P} = I$$

So $I - 2P$ is unitary matrix.

Actually geometric interpretation is $I - 2P$ preserves the norm like the figure below:



but as you see v and $(I-2p)v$ are two diagonals of rectangle so they have equal length.

problem #4: assume $A = [a_1 | a_2 | \cdots | a_n]$

then like (7.3) we have

$$a_1 = r_{11} q_1$$
$$a_2 = r_{12} q_1 + r_{22} q_2$$
$$a_3 = r_{13} q_1 + r_{23} q_2 + r_{33} q_3$$
$$\vdots$$

Claim $\hat{R} = \begin{bmatrix} r_{11} & 0 & r_{13} & 0 & r_{15} & \\ & r_{22} & 0 & r_{24} & 0 & \\ & & r_{33} & 0 & r_{35} & \\ & & & r_{44} & 0 & \\ & & & & r_{55} & \\ & & & & & \ddots \end{bmatrix}$

Since we have assumed $m \geq n$ and A has rank n so a_i are independent. and $\langle a_{2k}, a_{2k+1} \rangle = 0$ so $\langle a_1, a_2 \rangle = 0 \Rightarrow r_{11} r_{12} = 0 \Rightarrow r_{12} = 0$

$\langle a_2, a_3 \rangle = 0 \Rightarrow r_{22} r_{23} = 0 \Rightarrow r_{23} = 0$, $\langle a_1, a_4 \rangle = 0 \Rightarrow r_{11} r_{14} = 0 \Rightarrow r_{14} = 0$

$\langle a_3, a_4 \rangle = 0 \Rightarrow r_{33} r_{34} = 0 \Rightarrow r_{34} = 0$ (Since a_1, a_3 are independent so $r_{33} \neq 0$)

So we prove by induction $a_{2k} = r_{22k} q_2 + r_{42k} q_4 + \cdots$

$a_{2k+1} = r_{12k+1} q_1 + r_{32k+1} q_3 + \cdots$

assume we proved for $i < k$ we want to prove a_{2k}

$\langle a_1, a_{2k} \rangle = 0 \Rightarrow r_{11} r_{12k} = 0 \Rightarrow r_{12k} = 0$ now again by induction

assume $r_{2l+1\, 2k} = 0$ for $l < j$ then $\langle a_{2j+1}, a_{2k} \rangle = 0 \Rightarrow r_{2j+1,2j+1} r_{2j+1,2k} = 0$

$\Rightarrow r_{2j+1\, 2k} = 0$ so by induction all $r_{2l+1,2k} = 0$ and similarly all $r_{2k,2l+1} = 0$

So we proved the claim.

problem #5 :

(a) $\|x\|_\infty \overset{?}{\leq} \|x\|_2$ $\qquad x = (x_1, \ldots, x_m)$

assume $\max |x_i| = |x_1|$ then

$$\|x\|_2^2 = \sum_{i=1}^{m} x_i^2 \geq x_1^2 = |x_1|^2 = \|x\|_\infty^2 \Rightarrow \|x\|_2 \geq \|x\|_\infty$$

and for example for $(1, 0, \ldots, 0)$ equality holds.

(b) $\|x\|_2 \overset{?}{\leq} \sqrt{m}\, \|x\|_\infty$

again assume $\|x\|_\infty = |x_1|$ then

$$\|x\|_2^2 = x_1^2 + \cdots + x_m^2 \leq x_1^2 + x_1^2 + \cdots + x_1^2 = m x_1^2 = (\sqrt{m}\, \|x\|_\infty)^2 \Rightarrow \|x\|_2 \leq \sqrt{m}\, \|x\|_\infty$$

and for example for $(1, 1, \ldots, 1)$ equality holds.

(c) $\|A\|_\infty \overset{?}{\leq} \sqrt{n}\, \|A\|_2$

$\|A\|_2 = \sup \dfrac{\|Ax\|_2}{\|x\|_2}$ So if we find x such that $\sqrt{n} \dfrac{\|Ax\|_2}{\|x\|_2} \geq \|A\|_\infty$

then we are done take $x = (\pm 1, \ldots, \pm 1)$ then $\|x\|_2 = \sqrt{n}$

and $Ax = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \begin{bmatrix} \pm 1 \\ \vdots \\ \pm 1 \end{bmatrix} = \begin{bmatrix} \|r_1\|_1 \\ * \\ \vdots \\ * \end{bmatrix}$, assume $\|A\|_\infty = \|r_1\|_1$ and

choose $\pm$ in each coordinate of x such $\langle r_1, x \rangle = \|r_1\|_1$

So first coordinate of Ax is $\|r_1\|_1$ so $\|Ax\|_2 \geq \|r_1\|_1 = \|A\|_\infty$

So we are done.

and for example for $A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & O & & \\ 0 & & & & \end{bmatrix}$ obviously equality holds.

(d) $\|A\|_2 \overset{?}{\leq} \sqrt{m}\, \|A\|_\infty$

So if we show for any x $\dfrac{\|Ax\|_2}{\|x\|_2} \leq \sqrt{m}\, \|A\|_\infty$ then we are done

$$\|Ax\|_2^2 = \left\| \begin{bmatrix} r_1 \\ \hline r_2 \\ \hline \vdots \\ r_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\|_2^2 = \sum_{i=1}^{m} \langle r_i, x \rangle^2 \qquad (I)$$

again assume $\|A\|_\infty = \|r_1\|_1$ and $r_i = (r_i^1, r_i^2, \ldots, r_i^n)$

So (I): $\|Ax\|_2^2 = \sum_{i=1}^{m} \langle r_i, x \rangle^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} r_i^j x_j \right)^2$

$$m \|A\|_\infty \|x\|_2 = m \left(\sum_{j=1}^{n} |r_1^j| \right)^2 \left(\sum x_j^2 \right)$$

note that by assumption $\left(\sum |r_1^j| \right)^2 \geq \left(\sum |r_i^j| \right)^2$

and trivially $\left(\sum |r_i^j| \right)^2 \geq \left(\sum |r_i^j|^2 \right)$

by Cauchy–Schwartz $\left(\sum |r_i^j|^2 \right) \left(\sum x_j^2 \right) \geq \left(\sum_j r_i^j x_j \right)^2$

So: $m \left(\sum_{j=1}^{n} |r_1^j| \right)^2 \left(\sum x_j^2 \right) \geq \left(\sum_j |r_1^j| \right)^2 \left(\sum x_j^2 \right) + \left(\sum |r_2^j| \right)^2 \left(\sum x_j^2 \right) + \cdots + \left(\sum |r_m^j| \right)^2 \left(\sum x_j^2 \right)$

$$\geq \left(\sum |r_1^j|^2 \right) \left(\sum x_j^2 \right) + \cdots + \left(\sum |r_m^j|^2 \right) \left(\sum x_j^2 \right)$$

$$\geq \left(\sum r_1^j x_j\right)^2 + \text{---} + \left(\sum r_m^j x_j\right)^2$$

$$\geq \sum_i \left(\sum_j r_i^j x_j\right)^2 = \|Ax\|_2^2$$

$$\Longrightarrow \quad m\,\|A\|_\infty^2\,\|x\|_2^2 \geq \|Ax\|_2^2$$

So $\forall x$ $$\frac{\|Ax\|_2^2}{\|x\|_2^2} \leq m\,\|A\|_\infty^2 \Longrightarrow \quad \|A\|_2 \leq \sqrt{m}\,\|A\|_\infty$$

and trivially for example equality holds for $\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & & \ddots & \vdots \\ \vdots & \vdots & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix}$

Sketch of Bonus Problem :

(a) $F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} = \left[\text{reflection by line } y=x\right] \circ R_{\theta-\pi/2}$

$R_{\theta-\frac{\pi}{2}}$ is rotation over origin by $\theta-\frac{\pi}{2}$

$J = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R_{-\theta}$ so is rotation by θ clockwise.

(b) like the figure 10.1 in each step. We have to find an operator F such that $x = \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix} \longrightarrow Fx = \begin{bmatrix} \|x\| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

but obviously we can do it with sequence of two dimensional rotations

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_\ell \end{bmatrix}$, first apply appropriate J s.t $J\underbrace{\begin{bmatrix} x_{\ell-1} \\ x_\ell \end{bmatrix}}_{\vec{v}_1} = \begin{bmatrix} \|v_1\| \\ 0 \end{bmatrix}$

and then apply appropriate J such that $J\underbrace{\begin{bmatrix} x_{\ell-2} \\ \|v_1\| \end{bmatrix}}_{\vec{v}_2} = \begin{bmatrix} \|v_2\| \\ 0 \end{bmatrix}$

and repeat until you get $\begin{bmatrix} \|x\| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

c) each $J\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta\, a + \sin\theta\, a \\ -\sin\theta\, a + \cos\theta\, b \end{bmatrix}$ has 6 flops since each coordinate has 2 multiplication and on addition so to get $\begin{bmatrix} \|x\| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

we need $6(\ell-1)$ but like 10.8 each column need $4\ell-1$ so we need approximately 50% more flops.