Mathematics 104, Fall 2010: Assignment #6

Due: **Wednesday, November 17th**

*Instructions:* Please ensure that your answers are legible. Also make sure that all steps are shown – even for problems consisting of a numerical answer. Bonus problems cover advanced material and, while good practice, are *not* required and will *not* be graded.

**Problem #1.** Consider the matrix

\[ A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \]

a) By hand compute the SVD of \( A \). To do this it is useful to recall that all vectors in \( x \) with \( ||x||_2 = 1 \) are of the form

\[ x = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}. \]

b) Using the SVD determine the rank one matrix \( B \) that best approximates \( A \) in the Frobenius norm.

c) Compare how well \( B \) approximates \( A \) in the Frobenius norm with how well the rank one matrices

\[ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \]

approximate \( A \) in the Frobenius norm.

**Problem #2.** Exercise 4.4 of Lecture 4 of Trefethen-Bau.

**Problem #3.** Let \( A_1, A_2 \in \mathbb{C}^{m \times m} \) suppose that the left singular vectors of \( A_1 \) are \( \{u_1, \ldots, u_m\} \) and the right singular vectors are \( \{v_1, \ldots, v_m\} \) while the left singular vectors of \( A_2 \) are \( \{u_1, \ldots, u_m\} \) and the right singular vectors are \( \{v_1, \ldots, v_m\} \). Show that if for \( 1 \leq i \leq m \)

\[ u_i = v_i \]

then \( A_1A_2 = A_2A_1 \). That is if two square matrices have the same left and right singular vectors then they commute. (Hint: Think about the problem for diagonal matrices first, then use the SVD of \( A_1 \) and \( A_2 \)).

**Problem #4.** Suppose that \( A \) has the following (full) SVD

\[ A = \begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 & 3/5 & 4/5 \end{bmatrix} \]

a) Using only the SVD give an orthonormal basis of \( R(A) \).

b) Compute a (full) \( QR \) factorization of \( A \).

c) Using the \( QR \) factorization of \( A \) give an orthonormal basis of \( R(A) \). Does this agree with your answer in a)?

**Problem #5.** Let \( A \in \mathbb{C}^{m \times m} \) have singular values \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0 \).

a) Show that if \( \lambda = \mu \sigma_1 \) is an eigenvalue of \( A \) for \( |\mu| = 1 \) then \( ||A^2||_2 = ||A||_2^2 \). Recall \( \lambda \) is an eigenvalue of \( A \) when the matrix \( A - \lambda I \) is singular.

b) Assume in addition that \( \sigma_1 > \sigma_2 \) show that if \( ||A^2||_2 = ||A||_2^2 \) then \( \mu \sigma_1 \) is an eigenvalue of \( A \) for some \( \mu \) with \( |\mu| = 1 \). (Hint: Think about the proof of the uniqueness of singular vectors for the SVD)

**Bonus Problem.** Determine if the condition in Problem # 5 part b) that \( \sigma_1 > \sigma_2 \) is necessary. That is prove the result without this assumption or give a counter-example.