problem #1 : (a) We apply A to unit circle to find smallest and biggest

vectors : $\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} \cos\theta + 2\sin\theta \\ 2\sin\theta \end{bmatrix}}_{y} \Longrightarrow \|y\|_2^2 = 1 + 7\sin^2\theta + 2\sin 2\theta$

$\frac{\partial}{\partial\theta}\left(\|y\|_2^2\right) = 7\sin 2\theta + 4\cos 2\theta = 0 \Rightarrow \tan 2\theta = \frac{-4}{7}$ so extermums attained

when $\tan 2\theta = \frac{-4}{7}$ so $2\theta = k\pi + \alpha$ where $\tan\alpha = \frac{-4}{7}$ so for $k = 0, 1$

We take max and min and periodicity other k does not give different value

for $\|y\|_2^2$: $\theta = \frac{\alpha}{2}$: $\|y\|_2^2 = \frac{9}{2} - \frac{7}{2}\cos 2\theta + 2\sin 2\theta \simeq 8.53112887$

$\theta = \frac{\pi}{2} + \frac{\alpha}{2}$: $\|y\|_2^2 = 0.468871126$

$\|y\|_{max} = 2.92080963$, $\|y\|_{min} = 0.684741649$

$\frac{y_{min}}{\|y\|_{min}} \simeq \begin{bmatrix} -0.6617 \\ 0.7496 \end{bmatrix}$, $x_{min} \simeq \begin{bmatrix} -0.9664 \\ 0.2566 \end{bmatrix}$, $x_{max} = \begin{bmatrix} 0.2566 \\ -0.9664 \end{bmatrix}$

$\frac{y_{max}}{\|y\|_{max}} = \begin{bmatrix} 0.7496 \\ 0.6618 \end{bmatrix}$ So $\underbrace{A}_{A}\underbrace{\left[x_{min} \mid x_{max}\right]}_{V} = \underbrace{\left[\frac{y_{min}}{\|y\|_{min}} \;\middle|\; \frac{y_{max}}{\|y\|_{max}}\right]}_{U}\underbrace{\begin{bmatrix} \|y\|_{min} & \\ & \|y\|_{max} \end{bmatrix}}_{\Sigma}$

b) the best Frobenius approxiamation is $\sigma_1 u_1 v_1^*$ $\sigma_1 = \|y\|_{max}$

u_1 in U (in a way that I wrote is second column) so is $\frac{y_{max}}{\|y_{max}\|}$ and

and if Compute the inverse of V you will see $v_1^* = \begin{bmatrix} -0.2956 \\ -1.1133 \end{bmatrix}$

So $B = \begin{bmatrix} 2.1894 \\ 1.9329 \end{bmatrix} [-0.2956, -1.1133] = \begin{bmatrix} -0.6471 & 2.4374 \\ -0.5713 & -2.1518 \end{bmatrix}$

$\| A-B \|_F = \sqrt{\sigma_2^2} = \sigma_2 = \| y_{min} \| = 0.6847$

c) $\| A - A_1 \|_F = \| \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \|_F = 2\sqrt{2}$

$0.6847 < 1 < 2\sqrt{2}$

$\| A - A_2 \|_F = \| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \|_F = 1$

I hope I did not make Computational mistake!

problem #2 , no, not necessarily. We know that singular value of A is square root of eigenvalue of A^*A. So we try to find two matrices, A, B such that A^*A, B^*B have same eigenvalues.

take $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Longrightarrow A^*A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

take $B = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix} \Longrightarrow B^*B = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$

characteristic polynomials of A^*A and B^*B are both $\lambda^2 - 5\lambda + 4$ so they have same eigenvalue so A, B have same singular values.

but if $A = UBU^*$ then A and B should have same eigenvalue

but eigenvalues of A are $1, 2$ but eigenvalues of B are $\sqrt{2}, \sqrt{2}$

(or you can use trace to show they are not unitary similar).

---

problem #3 : Since $u_i^1 = v_i^1 = u_i^2 = v_i^2$ so in SVD of A_1 and A_2:

$$A_1 = U_1 \Sigma_1 V_1^{-1} \qquad A_2 = U_2 \Sigma_2 V_2^{-1}$$

We have $U_1 = V_1 = U_2 = V_2$ so $A_1 = U\Sigma_1 U^{-1}$ and $A_2 = U\Sigma_2 U^{-1}$

$A_1 A_2 = U\Sigma U^{-1} U \Sigma_2 U^{-1} = U\Sigma_1\Sigma_2 U^{-1}$, $A_2 A_1 = U\Sigma_2 U^{-1} U \Sigma_1 U^{-1} = U\Sigma_2\Sigma_1 U^{-1}$

assume $\Sigma_1 = \begin{bmatrix} \sigma_1^1 & & & 0 \\ & \sigma_2^1 & & \\ & & \ddots & \\ 0 & & & \sigma_m^1 \end{bmatrix}$, $\Sigma_2 = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_m^2 \end{bmatrix}$ where σ_i^1's are singular values

of A_1 and σ_j^2's are singular values of A_2

then obviously $\Sigma_1 \Sigma_2 = \Sigma_2 \Sigma_1 = \begin{pmatrix} \sigma_1^1 \sigma_1^2 & & & 0 \\ & \sigma_2^1 \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_m^1 \sigma_m^2 \end{pmatrix}$

So $A_1 A_2 = A_2 A_1$

oblem #4 :

(a) by definition of SVD $A = U\Sigma V^{-1}$ so U is a matrix of left singular vectors because in full SVD in Σ we have one zero row so the last column of U is added to make it full from reduced SVD so first two column of matrix left singular vectors are orthogonal basis for

$$R(A) = \operatorname{span}\left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \right\rangle$$

(b) since U is unitary so if we find QR factorization for $\Sigma V^{-1} = Q_1 R_1$ then QR factorisation of A would be $A = \underbrace{UQ_1}_{Q} R_1$.

$$\Sigma V^{-1} = \begin{array}{c} \begin{bmatrix} 8/5 & -6/5 \\ 3/5 & 4/5 \\ 0 & 0 \end{bmatrix} \\ \begin{matrix} a_1 \quad & \quad a_2 \end{matrix} \end{array}$$

$$a_1 = r_{11} q_1 = \frac{5}{\sqrt{73}} \cdot \frac{\sqrt{73}}{5}\left(\frac{8}{5}, \frac{3}{5}, 0\right) = \underbrace{\frac{\sqrt{73}}{5}}_{r_{11}} \underbrace{\left(\frac{5}{\sqrt{73}}\left(\frac{8}{5}, \frac{3}{5}, 0\right)\right)}_{q_1}$$

so q_2 is unit vector orthogonal to q_1 and has zero as last coordinate take $q_2 = \frac{1}{\sqrt{73}}(-3, 8, 0)$ then,

$$a_2 = \underbrace{\frac{-36}{\sqrt{73}}}_{r_{12}} \cdot \underbrace{\left(\frac{1}{\sqrt{73}}(8,3,0)\right)}_{q_1} + \underbrace{\frac{50}{\sqrt{73}}}_{r_{22}} \underbrace{\left(\frac{1}{\sqrt{73}}(-3,8,0)\right)}_{q_2}$$

So

$$A = \underbrace{\begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 8/\sqrt{73} & -\frac{3}{\sqrt{73}} \\ 3/\sqrt{73} & 8/\sqrt{73} \\ 0 & 0 \end{bmatrix}}_{\hat{Q}} \underbrace{\begin{bmatrix} \sqrt{73}/5 & \frac{-36}{\sqrt{73}} \\ 0 & \frac{50}{\sqrt{73}} \end{bmatrix}}_{\hat{R}}$$

which gives reduced QR factorization for full add zero row to $\hat{R}$

and find arbitrary unit orthogonal vector to two column of $\hat{Q}$.

c) $\hat{Q} = \begin{bmatrix} \frac{3\sqrt{2}}{2\sqrt{73}} & \frac{4\sqrt{2}}{\sqrt{73}} \\ 8/\sqrt{73} & \frac{-3}{\sqrt{73}} \\ \frac{-3\sqrt{2}}{2\sqrt{73}} & \frac{-4\sqrt{2}}{\sqrt{73}} \end{bmatrix}$ with columns q_1, q_2, so $Q = \left[q_1 \,\middle|\, q_2 \,\middle|\, \begin{matrix} \sqrt{2}/2 \\ 0 \\ \frac{\sqrt{2}}{2} \end{matrix} \right]$

q_1, q_2 are orthogonal vectors that are basis for $R(A)$ and is

different orthogonal basis for $R(A)$.

problem #5

a) from (3.14) we know that $\|A^2\|_2 \le \|A\|_2^2 = \sigma_1^2$ so for any unit vector u $\|A^2 u\|_2 \le \sigma_1^2$ so if we find unit vector u such that

$A^2 u = \mu^2 \sigma_1^2 v$ where $|\mu| = 1$, $\|v\|_2 = 1$ $\Longrightarrow$ $\|A^2\|_2 \ge \sigma_1^2 \Longrightarrow \|A^2\|_2 = \sigma_1^2$

but by hypothesis we know that $\exists$ unit vector u s.t $Au = \mu \sigma_1 u$

So $A(Au) = \mu \sigma_1 Au = \mu^2 \sigma_1^2 u \Longrightarrow \|A\|_2 \ge \sigma_1^2$ so we are done

(b) So $\exists$ v_1 s.t $A^2 v_1 = \sigma_1^2 u$, v_1 cannot be eigenvalue of A if

we assume $\mu \sigma_1$ is not eigenvalue of A. Since if $Av_1 = \lambda v_1$ then $A^2 v_1 = \lambda^2 v_1$

so $\lambda = \mu \sigma_1$ for some μ, $|\mu| = 1$. so Av_1, v_1 are linearly independent take unit

vector u orthogonal to v_1, Av_1 and $u \in \text{Span}(v_1, Av_1)$

Claim $\langle Au, Av_1 \rangle = 0$

Since $\langle Au, Av_1 \rangle = \langle u, A^* A v_1 \rangle = \langle u, \sigma_1^2 v_1 \rangle = 0$

now write $Av_1 = cu + sv_1$ since $u \perp v_1$ we have $|c|^2 + |s|^2 = \|Av_1\|_2^2 \le \sigma_1^2$

$A^2 v_1 = cAu + sAv_1$ since $Au \perp Av_1 \Longrightarrow \sigma_1^4 = \|A^2 v_1\|_2^2 = |c|^2 \|Au\|_2^2 + |s|^2 \|Av_1\|_2^2$

so both $\|Av_1\|_2^2 = \|Au\|_2^2 = \sigma_1^2$ since otherwise in general $\|Av_1\|_2^2 \le \sigma_1^2$ $\|Au\|_2^2 \le \sigma_1^2$

$|c|^2 \|Au\|_2^2 + |s|^2 \|Av_1\|_2^2 \le |c|^2 \sigma_1^2 + |s|^2 \sigma_1^2 = \sigma_1^2 (|c|^2 + |s|^2) \le \sigma_1^4$ so in order to have equality both should be σ_1^2 but since $\sigma_1 > \sigma_2$ by uniqueness singular vector

correspondent to σ_1 is unique but $u_1 v_1$ are different vectors so contradiction

so $\rho\sigma_1$ is eigenvalue of A.

Bonus Problem

no $\sigma_1 > \sigma_2$ is essential. take $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. A and A^2 are unitary

matrices. so $\|A^2\|_2 = \|A\|_2^2 = 1$ and since $A^*A = I$ so $\sigma_1 = \sigma_2 = 1$ if $\theta \neq 2k\pi$

1 is not eigenvalue of A.