problem #1

(a) $Sv = \lambda v$ take v be unit eigen vector so:

$$\bar{\lambda} = \bar{\lambda}(v,v) = (\lambda v, v) = (Sv, v) = (v, S^* v) = (v, -Sv) = (v, -\lambda v) = -\lambda(v,v) = -\lambda$$

$\Longrightarrow$ $\lambda + \bar{\lambda} = 0 \Longrightarrow \lambda$ is purely imaginary

(b) if not then $\exists\, v$ such that $(I-S)v = 0 \Longrightarrow Sv = v \Longrightarrow 1$ is eigen value

but by it has to be pure imaginary so contrary implies $I-S$ is invertible.

(c) first note that by definition easily deduced $(A^T)^{-1} = (A^{-1})^T$ for any invertible A. now since $(I-S)$ and $(I+S)$ commute we have:

$$(I-S)^{-1}(I+S) = (I-S)^{-1}\underbrace{(I+S)(I-S)}(I-S)^{-1} = (I-S)^{-1}(I-S)(I+S)(I-S)^{-1} = (I+S)(I-S)^{-1}$$

so $I+S$ and $(I-S)^{-1}$ commute now:

$$\left((I-S)^{-1}(I+S)\right)^{-1} = (I+S)^{-1}(I-S)$$

$$\left((I-S)^{-1}(I+S)\right)^{*} = (I+S)^{*}\left((I-S)^{-1}\right)^{*} = (I+S)^{*}\left((I-S)^{*}\right)^{-1} = (I+S^{*})(I-S^{*})^{-1}$$

$= (I-S)(I+S)^{-1}$ but similarly $(I+S)^{-1}(I-S) = (I-S)(I+S)^{-1}$ since

$$(I+S)^{-1}(I-S) = (I+S)^{-1}\underbrace{(I-S)(I+S)}(I+S)^{-1} = (I+S)^{-1}(I+S)(I-S)(I+S)^{-1} = (I-S)(I+S)^{-1}$$

$$\Longrightarrow \left((I-S)^{-1}(I+S)\right)^{-1} = \left((I-S)^{-1}(I+S)\right)^{*} \Longrightarrow (I-S)^{-1}(I+S) \text{ is unitary.}$$

problem #2 : Since by definition $\|A\| = \sup_{x \in \mathbb{C}^m} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$

so take unit eigenvector which has absolute value $\rho(A)$, let say λ.

$Av = \lambda v \quad \|v\|=1 \quad |\lambda| = \rho(A)$ so $\|A\| \geq \|Av\| = \|\lambda v\| = |\lambda| \|v\| = |\lambda| = \rho(A)$

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problem #3 We know singular values of A are square root of AA^* :

$$AA^* = \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \\ -3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 2 \\ 2 & 6 \end{bmatrix}$$

$$P_\lambda(AA^*) = (14-\lambda)(6-\lambda) - 4 = \lambda^2 - 20\lambda + 80$$

$$\lambda_1 = +10 + 2\sqrt{5} \qquad \lambda_2 = +10 - 2\sqrt{5}$$

so $\sigma_1 = \sqrt{10 + 2\sqrt{5}} \qquad \sigma_2 = \sqrt{10 - 2\sqrt{5}}$

problem #4 : $A = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$

$$P_\lambda(A) = (r\cos\theta - \lambda)(r\cos\theta - \lambda) + r^2\sin^2\theta = \lambda^2 - 2r\cos\theta\,\lambda + r^2 = 0$$

$$\lambda = \frac{2r\cos\theta \pm \sqrt{4r^2\cos^2\theta - 4r^2}}{2} = r\cos\theta \pm ir|\sin\theta| \quad \text{take} \quad \begin{matrix} \lambda_1 = r\cos\theta + ir\sin\theta \\ \lambda_2 = r\cos\theta - ir\sin\theta \end{matrix}$$

$$(A - \lambda_1 I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow \begin{bmatrix} ir\sin\theta & -r\sin\theta \\ r\sin\theta & ir\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow x_1 = ir\sin\theta,\ x_2 = -r\sin\theta$$

So $v_1 = \begin{bmatrix} ir\sin\theta \\ -r\sin\theta \end{bmatrix}$ is eigenvector (and scalar dilation of v_1) for λ_1

$$(A - \lambda_2 I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow \begin{bmatrix} -ir\sin\theta & -r\sin\theta \\ r\sin\theta & -ir\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow \quad x_1 = -ir\sin\theta \quad x_2 = -r\sin\theta$$

So $v_2 = \begin{bmatrix} -ir\sin\theta \\ -r\sin\theta \end{bmatrix}$ is eigenvector (and any scalar dilation) for λ_2

problem #5

(a) $A = \begin{bmatrix} 13 & 9 \\ -16 & 37 \end{bmatrix}$ $\quad B = \begin{bmatrix} -35 & 45 \\ 20 & 10 \end{bmatrix}$

$P_\lambda(A) = (\lambda - 25)^2 \qquad P_\lambda(B) = (\lambda+50)(\lambda-25)$

you can easily check $\frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is unit eigenvector for A

so take $U = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$ then

$$U^* A U = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}\begin{bmatrix} 13 & 9 \\ -16 & 37 \end{bmatrix}\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 25 & 25 \\ 0 & 25 \end{bmatrix}$$

$$A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}\begin{bmatrix} 25 & 25 \\ 0 & 25 \end{bmatrix}\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$$

again you can easily check $\frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is unit eigenvector for $\lambda = 25$ for B. so again take $U = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$

$U^* B U = \begin{bmatrix} 25 & 25 \\ 0 & -50 \end{bmatrix}$ so $B = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}\begin{bmatrix} 25 & 25 \\ 0 & -50 \end{bmatrix}\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$

(b) A is not diagonlizable since other wise $A = X\Lambda X^{-1}$ where Λ is diagonal matrix which has same characteristic polynomial so Λ has to be

$\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = 25I$ Then $A = X\Lambda X^{-1} = X(25I)X^{-1} = 25XIX^{-1} = 25I$ $\times$

So A is not diagonalizeble but B is diagonalizeble we know eigenvector for $\lambda = 25$ is $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and for $\lambda = -50$ is $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

So take $X = \begin{bmatrix} 3 & 3 \\ 4 & -1 \end{bmatrix}$ then B in basis of eigenvector is in diagonal form.

$$X^{-1} B X = \frac{-1}{15} \begin{bmatrix} -1 & -3 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -35 & 45 \\ 20 & 10 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 4 & -1 \end{bmatrix} =$$

$$\frac{-1}{15} \begin{bmatrix} -1 & -3 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 75 & -150 \\ 100 & 50 \end{bmatrix} = \frac{-1}{15} \begin{bmatrix} -375 & 0 \\ 0 & 750 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & -50 \end{bmatrix}$$

So $B = \begin{bmatrix} 3 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 25 & 0 \\ 0 & -50 \end{bmatrix} \begin{bmatrix} \frac{1}{15} & \frac{1}{5} \\ \frac{4}{15} & -\frac{1}{5} \end{bmatrix}$

Bonus Problem: By schuro factorization we know A is unitary triangulizable

So assume $A = QTQ^*$ where Q is unitary and T is triangular.

then obviously $AA^* \overset{?}{=} A^*A$ implies $TT^* \overset{?}{=} T^*T$ but T is triangular

so $[TT^*]_{11} = \sum_i |t_{1i}|^2$ $\quad [T^*T]_{11} = |t_{11}|^2$ so $t_{12} = t_{13} = \cdots = t_{1n} = 0$

now we get the first row except t_{11} has zero entry so by induction we are done.