Math 104 : Midterm

Instructions: Complete the following 4 problems. Remember to show all your work. No notes or calculators are allowed. Please sign below to indicate you accept the honor code.

Name: ____________________________

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Problem #1. (20 pts) Let $w_1$, $w_2$ and $w_3$ be three vectors in $\mathbb{C}^3$. Let
\begin{align*}
v_1 &= w_1 - w_3, \\
v_2 &= w_1 + w_2, \\
v_3 &= w_1 + \lambda w_3, \text{ and} \\
v_4 &= 2w_1 + w_2 - w_3.
\end{align*}
Where here $\lambda \in \mathbb{C}$. For what value $\lambda_0$ is it always true that when $\lambda = \lambda_0$, $v_1, v_2, v_3$ and $v_4$ never span $\mathbb{C}^3$. Justify your answer. (Hint: Rewrite the problem using matrices).

Answer:
Let us set 
\begin{align*}
V &= [v_1 | v_2 | v_3 | v_4] \\
W &= [w_1 | w_2 | w_3]
\end{align*}
Then we have 
\begin{align*}
V &= WA
\end{align*}
where 
\begin{align*}
A &= \begin{bmatrix}
1 & 1 & 1 & 2 \\
0 & 1 & 0 & 1 \\
-1 & 0 & \lambda & -1
\end{bmatrix}
\end{align*}
The $v_i$ do not span $\mathbb{C}^3$ when and only when $\dim R(V) \leq 2$. By the rank-nullity theorem this occurs when and only when $\dim N(V) \geq 1$. Notice that $N(A) \subset N(V)$ and so $\dim N(V) \geq \dim N(A)$. Applying the Gaussian elimination algorithm to $A$ one arrives after a sequence of row operations to $A'$ with 
\begin{align*}
A' &= \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & \lambda + 1 & 0
\end{bmatrix}
\end{align*}
Notice if $\lambda + 1 = 0$ then $rref(A)$ has 2 pivots. Otherwise $rref(A)$ has 3 pivots. In the former case, $\dim N(A) = \dim N(rref(A)) = 4 - 2 = 2$ while in the latter $\dim N(A) = \dim N(rref(A)) = 4 - 3 = 1$. In particular, $\lambda_0 = -1$ always ensures that the $v_i$ do not span. Notice that if the $w_i$ are linearly independent and $\lambda_0 \neq -1$ then $\dim N(V) = \dim N(A) = 1$ and so $\dim R(V) = 3$. In particular, the $v_i$ would span $\mathbb{C}^3$ in this case.
Problem #2. (30 pts) Let 
\[ v = \begin{bmatrix} 2 \sin \theta \\ -2 \cos \theta \end{bmatrix} \in \mathbb{R}^2 \]

Let \( A \in \mathbb{R}^{2 \times 2} \) denote the matrix which gives orthogonal projection onto \( \text{span}(v) \).

a) Determine \( A \).

Answer:
As \( v \neq 0 \), \( \text{span}(v) \) is one dimensional. Hence there is a unit length vector \( q \in \text{span}(v) \) and \( \text{span}(q) = \text{span}(v) \). In particular, take 
\[ q = \frac{1}{2} v = \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix}. \]

The projector \( A \) is then given by:
\[ A = qq^* = \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix} = \begin{bmatrix} \sin^2 \theta & -\sin \theta \cos \theta \\ -\sin \theta \cos \theta & \cos^2 \theta \end{bmatrix}. \]

b) Determine \( N(A) \) and \( R(A) \).

Answer:
As \( A \) is a projector onto \( \text{span}(v) \) one must have \( R(A) = \text{span}(v) \) by definition. Because \( A \) is an orthogonal projector one also has \( R(A) = N(A) \). Notice that for 
\[ w = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \]

one has \( w \neq 0 \) and \( \langle w, v \rangle = 0 \). So \( w \in R(A) = N(A) \) and is non-trivial. As \( \text{dim}N(A) = 1 \) (by the rank-nullity theorem for instance) one then has \( N(A) = \text{span}(w) \).
c) Determine a full $QR$ factorization of $A$.

**Answer:**
The first column of $A$ is

\[ a_1 = \begin{bmatrix} \sin^2 \theta \\ -\sin \theta \cos \theta \end{bmatrix} = \sin \theta \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix} = \sin \theta q_1. \]

Here $q_1$ is a unit vector and $\text{span}(a_1) \subset \text{span}(q_1)$. Hence is the correct choice for the $QR$ factorization. In this case $r_{11} = \sin \theta$. We now set

\[ r_{12} = \langle q_1, a_2 \rangle = -\sin^2 \theta \cos \theta - \cos^3 \theta = -\cos \theta (\sin^2 \theta + \cos^2 \theta) = -\cos \theta \]

Then with $a_2$ the second column of $A$

\[ \hat{q}_2 = a_2 - r_{12}q_1 = 0 \]

Thus we just need to pick $q_2$ orthogonal to $q_1$ and set $r_{22} = 0$ we determined such a $q_2$ in b) Hence we have the full $QR$ factorization

\[ A = QR = \begin{bmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \sin \theta & -\cos \theta \\ 0 & 0 \end{bmatrix} \]

We note that the $Q$ term is unitary (as the columns are orthonormal) and the $R$ term is upper-triangular so this is indeed a $QR$ factorization.
Problem #3. (20 pts) Let $A, B \in \mathbb{C}^{m \times m}$ suppose that $AB = 0$ and $BA = 0$.

a) What, if any, is the relationship between the null space of $A$ and the column space of $B$? Justify your answer.

Answer:
One has $R(B) \subset N(A)$ as if $v \in R(B)$ then there is a $w$ so that $v = Bw$ and then $Av = ABw = 0w = 0$ and so $v \in N(A)$. There is no other relationship since for instance if $A = 0$ and $B = I$ then $AB = 0 = BA$ but $R(B) = \mathbb{C}^m$ while $R(A) = \{0\}$.

b) Show that either $\dim N(A) \geq \frac{m}{2}$ or $\dim N(B) \geq \frac{m}{2}$.

Answer:
By part a) we have that $R(B) \subset N(A)$. Hence by the basis extension theorem we have $\dim R(B) \leq \dim N(A)$. By the rank-nullity theorem applied to $B$ $\dim R(B) + \dim N(B) = m$. Thus,

$$m = \dim R(B) + \dim N(B) \leq \dim N(B) + \dim N(A)$$

This implies either $\dim N(B)$ or $\dim N(A)$ is larger than $m/2$ as claimed.
Problem #4. (30 pts)

a) Suppose that $A, B \in \mathbb{C}^{m \times m}$ are unitary matrices. Verify that $A^*$ and $AB$ are also unitary. (Hint: Use the algebraic properties of the adjoint)

Answer:
Since $A$ is unitary one has $AA^* = I$ taking the adjoint of this implies $(AA^*)^* = I^* = I$. Thus $(A^*)^* A^* = I$ that is $(A^*)^{-1} = (A^*)^*$ which means $A^*$ is unitary. Similarly, $BB^* = I$ so $(AB)^* AB = B^* A^* AB = B^* IB = B^* B = I$ which implies $(AB)^{-1} = (AB)^*$ and so $AB$ is unitary.

b) Let
\[
\mathbf{v}_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \mathbf{v}_2 = \frac{1}{5} \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3
\]
Verify that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Justify your answer.

Answer:
By direct computation one verifies that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$ (i.e. is 1 when $i = j$ and otherwise 0). This implies the set is orthonormal. Any orthogonal set of vectors is automatically linearly independent and as there are three elements in the set and $\dim \mathbb{C}^3 = 3$ we must have that the $\mathbf{v}_i$ also span $\mathbb{C}^3$ and hence are a basis.
c) Let
\[ w_1 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, w_3 = \frac{\sqrt{2}}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \]
be a set of orthonormal vectors. Determine the orthogonal matrix \( U \in \mathbb{R}^{3 \times 3} \) so that \( U v_i = w_i \) for \( i = 1, 2, 3 \) here the \( v_i \) are given in b). (Hint: Use part a )

**Answer:**
Let us write
\[ V = [v_1 | v_2 | v_3] \quad \text{and} \quad W = [w_1 | w_2 | w_3]. \]
As both \( \{v_i\} \) and \( \{w_i\} \) are orthonormal bases of \( \mathbb{C}^3 \) both \( V \) and \( W \) are orthogonal matrices. We note that orthogonal and unitary are the same in this context as both matrices have real entries. Then we have
\[ W = UV \]
so
\[ U = W V^{-1} = W V^\top (= W V^*) \]
By part a) \( U \) is unitary and has real entries so is orthogonal. We compute explicitly
\[
U = \begin{bmatrix}
3\sqrt{2}/10 & -\sqrt{2}/2 & -2\sqrt{2}/5 \\
-4/5 & 0 & -3/5 \\
3\sqrt{2}/10 & \sqrt{2}/2 & -2\sqrt{2}/5
\end{bmatrix}
\]