Math 104 - Fall 2008 - Final Exam

Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

Instructions: Print your name and student ID number, write your signature to indicate that you accept the honor code. During the test, you may not use computers, phones, or any other electronic device. Read each question carefully, and show all your work. Justify all your answers. You have three hours (8:30AM-11:30AM) to answer all the questions.

Question	Score	Maximum
1		15
2		10
3		10
4		15
5		10
6		10
7		15
8		10
9		5
Total		100

Problem 1. In a fictional soccer match-up between the national teams of Argentina (A), Belgium (B), and Croatia (C), the following results were recorded:

$$A \text{ vs. } B : 2-0 \qquad\qquad A \text{ vs. } C : 3-1 \qquad\qquad B \text{ vs. } C : 0-1.$$

We form a matrix M with the ratios "team's score" over "total score",

$$M = \begin{pmatrix} 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \end{pmatrix}.$$

For instance, the element $(1,3)$ or (A,C) is equal to $\frac{3}{4}$ because A scored 3 of the 4 goals in the A vs. C match. We arbitrarily chose to put zeros on the diagonal.

Consider the eigenvector v corresponding to the largest (real-valued) eigenvalue of M. The components v_A, v_B, v_C of this eigenvector are indicators of the performance of the respective teams: the higher the number, the better the team. So these numbers provide a pretty good, automatic way of ranking the teams.

(a) (5 pts.) Compute the eigenvalues of M.

(b) (5 pts.) Compute the normalized eigenvector v of M corresponding to the largest eigenvalue of M. Check that the values obtained for the components of v obey $v_A > v_C > v_B$, hence reproduce the expected ranking "A is better than C, and C is better than B".

(c) (5 pts.) Consider M defined with some other number than zero on the diagonal, say

$$M_\mu = \begin{pmatrix} \mu & 1 & \frac{3}{4} \\ 0 & \mu & 0 \\ \frac{1}{4} & 1 & \mu \end{pmatrix}.$$

Would this alternative choice have changed the ranking of the teams? In your answer, justify how μ affects the eigenvalues and eigenvectors of M_μ compared to those of M.

Problem 2. (10 pts.) Prove that the product of two n-by-n upper-triangular matrices is still upper-triangular, by induction over n. [Hint: for the induction step, decompose each $(n+1)$-by-$(n+1)$ matrix into four blocks of your choosing. For matrix-matrix multiplication, the rule of going along rows of the first matrix and down columns of the second matrix also works with blocks when their sizes properly match.]

Problem 3.

(a) (6 pts.) Fill in with any choice of “range space”, “row space”, “nullspace”, or “left-nullspace”, such that the following sentences are true:

- The relation $A^2 = 0$ implies that the ________________________ of A is included in the nullspace of A.
- The nullspace is the orthogonal complement of the ________________________.
- The dimension of the ________________________ is equal to the dimension of the ________________________, and is equal to the rank.

(b) (4 pts.) Prove that if A is an n-by-n matrix such that $A^2 = 0$, then necessarily $\text{rank}(A) \leq n/2$. [Hint: use all three assertions of part (a).]

Problem 4. Let $S \in \mathbb{C}^{m \times m}$ be skew-hermitian, i.e., $S^* = -S$.

(a) (5 pts.) Show that the eigenvalues of S are pure imaginary, i.e., are complex numbers with the real part equal to zero. [Hint: contrast with the situation of hermitian matrices $A^* = A$, as in problem 2 of Homework 4.]

(b) (5 pts.) Show that $I - S$ is invertible. [Hint: among the various criteria for invertibility, consider the one involving eigenvalues.]

(c) (5 pts.) Show that the matrix $Q = (I - S)^{-1}(I + S)$, known as the Cayley transform of A, is unitary.

Problem 5.

(a) (5 pts.) Prove that the eigenvalues of a projector P can have no other value than zero or one.

(b) (5 pts.) Prove that the singular values of an *orthogonal* projector P can have no other value than zero or one.

Problem 6. (10 pts.) Prove the inequality

$$\|A\|_F \leq \sqrt{\operatorname{rank}(A)}\, \|A\|_2.$$

[Hint: use the SVD.]

Problem 7. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(a) (8 pts.) Compute the reduced QR decomposition $A = \hat{Q}\hat{R}$. [Hint: use Gram-Schmidt to obtain $\hat{Q}$, and then use $\hat{R} = \hat{Q}^* A$ to obtain $\hat{R}$.]

(b) (7 pts.) Using your result in part (a), solve $Ax = b$ in the least squares sense, where

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 8.

(a) (5 pts.) Assume that A is an invertible square matrix. Let $\|A\|$ be any induced norm of A. Prove that the condition number

$$\kappa(A) = \|A\| \, \|A^{-1}\|$$

is always greater than or equal to one. [Hint: don't specialize to a special choice of norm, just use properties of induced norms.]

(b) (5 pts.) Compute $\kappa_\infty(A)$ when the norm is chosen as the infinity norm, and

$$A = \begin{pmatrix} 1 & 2 \\ \\ 3 & 4 \end{pmatrix}, \qquad\qquad A^{-1} = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ \\ 3 & -1 \end{pmatrix}.$$

Verify that indeed $\kappa_\infty(A) \geq 1$.

Problem 9. (5 pts.) Let $A, B \in \mathbb{R}^{m\times n}$, with $m \geq n$, such that $\text{range}(A) = \text{range}(B)$. Show that B^*A is invertible if and only if A is full column-rank. [Hint: contrast with problem 5 of Homework 5.]