1. Consider the three vectors

\[ v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}. \]

(a) Find the dimension of the vector space generated by \( v_1, v_2, v_3 \).
(b) Do \( v_1, v_2, v_3 \) form a basis for the space they generate?
(c) Find a matrix, whose entries are not all zeros, and whose nullspace contains all three vectors \( v_1, v_2, v_3 \).
(d) Find a vector \( w \) such that \( \{ v_1, v_2, w \} \) is a basis for \( \mathbb{R}^3 \).

2. Justify the statement: if a collection of vectors contains the zero vector, then there is no chance that the vectors in the collection be linearly independent.

3. Two vectors are said to be collinear when one can be written as a scalar multiple of the other. Consider two vectors \( u \) and \( v \) that are not collinear. Consider a vector \( w \) that does not belong to the linear span of \( u \) and \( v \). Prove that \( u, v, w \) are linearly independent.

4. A matrix is said to be upper triangular if \( a_{ij} = 0 \) for \( i > j \). Consider a generic 3-by-3 upper triangular matrix

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}. \]

(a) If \( a_{11}, a_{22}, \) and \( a_{33} \) are nonzero, show that the only solution to \( Ax = 0 \) is \( x = 0 \).
(b) If either \( a_{11} = 0 \), or \( a_{22} = 0 \) or \( a_{33} = 0 \), then prove that the columns are linearly dependent. (Consider all three cases separately.)
(c) If \( a_{22} = 0 \), find a nonzero element in the nullspace of \( A \).