Solutions Midterm Exam — Jul. 17, 2019

1. (10 points) Evaluate the following limit. You may use any technique as long as you justify your steps.

$$\lim_{x\to 0} \frac{(1-\sqrt{1+x})\sin(-2x)}{x^2}.$$

Observe that, for $x \neq 0$, (and $x > -1$)

$$\frac{1-\sqrt{1+x}}{x} = \frac{(1-\sqrt{1+x})}{x}\frac{(1+\sqrt{1+x})}{(1+\sqrt{1+x})} = \frac{1-(1+x)}{x(1+\sqrt{1+x})} = \frac{-x}{x(1+\sqrt{1+x})} = -\frac{1}{1+\sqrt{1+x}}$$

Hence, by the limit laws (alternatively because $\frac{1}{1+\sqrt{1+x}}$ is continuous at $x = 0$) we have

$$\lim_{x\to 0} \frac{1-\sqrt{1+x}}{x} = -\frac{1}{2}.$$

If $g(x) = \sin(-2x)$, then, as $g(0) = 0$

$$\lim_{x\to 0} \frac{\sin(-2x)}{x} = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = g'(0) = -2$$

where we used $g'(x) = -2\cos(2x)$. Hence, by the product law

$$\lim_{x\to 0} \frac{(1-\sqrt{1+x})\sin(-2x)}{x^2} = \left(\lim_{x\to 0} \frac{1-\sqrt{1+x}}{x}\right)\left(\lim_{x\to 0} \frac{\sin(-2x)}{x}\right) = (-\frac{1}{2}) * (-2) = 1.$$

2. (10 points) Let f be a function so that $f(2) = -1$, $f'(2) = -1$ and $f'(-1) = 2$. If

$$h(x) = f(f(x)) - \frac{f(x)}{x^2}$$

determine $h'(2)$.

Using the rules of differentiation (namely the chain rule, difference rule and quotient rule) we have

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(f(f(x)) - \frac{f(x)}{x^2}\right) \\ &= f'(f(x))f'(x) - \frac{f'(x)x^2 - 2xf(x)}{x^4} \end{aligned}$$

Evaluating at $x = 2$ gives

$$h'(2) = f'(f(2))f'(2) - \frac{4f'(2) - 4f(2)}{16} = -f'(-1) - \frac{-1-(-1)}{4} = -2.$$

3. (10 points) Use the linearization at $x = 1$ to compute the approximate value of $\ln(1.1)$.

Let $L(x)$ be the linearization of $f(x) = \ln(x)$ at $x = 1$. By definition we have that $L(x) = f'(1)(x-1) + f(1)$. One computes that $f'(x) = \frac{1}{x}$ and so $f'(1) = 1$ while $f(1) = \ln(1) = 0$. Hence, $L(x) = (x-1) + 0 = x - 1$. It follows that $\ln(1.1) = f(1.1) \approx L(1.1) = 0.1$.

4. (10 points) Consider the curve defined implicitly by $e^{2y} + x^2 = 2\cos(y^2)$. Determine the tangent line to this curve at $(x, y) = (1, 0)$.

First observe that $e^{2*0} + 1^2 = 1 + 1 = 2 = 2\cos(0^2)$ so $(1, 0)$ is on the curve. Next we differentiate implicitly and obtain that

$$\frac{d}{dx}\left(e^{2y} + x^2\right) = \frac{d}{dx}\left(2\cos(y^2)\right)$$

evaluating both sides gives

$$2e^{2y}\frac{dy}{dx} + 2x = -4\sin(y^2)y\frac{dy}{dx}.$$

Evaluating this expression at $(1, 0)$ gives

$$2\frac{dy}{dx}|_{(1,0)} + 2 = 0$$

and so

$$\frac{dy}{dx}|_{(1,0)} = -1.$$

That is the slope of the tangent line at $(1, 0)$ is -1. As the tangent line goes through the point $(1, 0)$ we have the equation of this line given by

$$y - 0 = -1(x - 1)$$

That is $y = 1 - x$.

5. (10 points) Find a and b so that the line $y = 5x + 3$ is tangent to $y = f(x) = x^3 + ax + b$ at $(-1, -2)$.

In order for $y = 5x + 3$ to be the tangent line to $y = f(x)$ at $(-1, -2)$ one must have $-2 = f(-1)$ and $5 = f'(-1)$. That is, the graph of f goes through $(-1, -2)$ and the slope of the tangent line is the slope of the given line. The first equation implies that $-2 = (-1)^3 + a(-1) + b$ and so $b - a = -1$. As $f'(x) = 3x^2 + a$ the second equation implies that $5 = 3(-1)^2 + a$ and so $a = 2$. Hence, $b = a - 1 = 1$ and so the answer is that $y = x^3 + 2x + 1$ is tangent to $y = 5x + 3$ at $(-1, -2)$.

6. Write the correct choice in the space provided. You do not need to show work. No partial credit.

(a) (5 points) What is the *largest* domain on which $f(x) = \ln(e^{-x} - 1)$ is defined?

A. $(0, \infty)$
B. $(-\infty, \infty)$
C. $\{x \neq 0\}$
D. $(-\infty, 0)$

(a) **D**

(b) (5 points) Determine the inverse of the function given by $f(x) = 4 - x^2$ on $[-2, 0]$.

A. $f^{-1}(y) = \sqrt{4-y}$ with domain $[0, 4]$
B. $f^{-1}(y) = -\sqrt{4-y}$ **with domain** $[0, 4]$
C. $f^{-1}(y) = \sqrt{4-y}$ with domain $[-4, 0]$
D. $f^{-1}(y) = -\sqrt{4-y}$ with domain $[-4, 0]$

(b) **B**

(c) (5 points) For what value c is $f(x) = \begin{cases} 2x - c & x < 1 \\ c - x^2 & x \geq 1 \end{cases}$ continuous at $x = 1$?

A. $c = 3$
B. $c = \frac{3}{2}$
C. $c = 0$
D. $c = -\frac{1}{2}$

(c) **B**

(d) (5 points) Suppose $f : [-5, 5] \to [-5, 5]$ is a continuous function for which $f(-1) = 2$, $f(0) = 3$ and $f(3) = -1$. In which interval *must* f have a zero?

A. $[-1, 3]$
B. $[-5, 0]$
C. $(3, 4)$
D. $[1, 4]$

(d) **A**

(e) (5 points) What is $\lim_{x\to 0} \frac{\sin(2x)}{x^2}$?

A. 0
B. 2
C. Does not exist
D. -2

(e) **C**

(f) (5 points) Which of the following functions is *not* differentiable at $x = 1$?

A. $f(x) = \ln(x)$

B. $f(x) = \frac{x}{1+x^2}$

C. $f(x) = \left|x^2 - 1\right|$

D. $f(x) = \ln(\cos(2\pi x))$

(f) **C**

(g) (5 points) What is the equation for the tangent line to $y = x^3 - x^2 - 3$ at $(2, f(2))$?

A. $y = 8x + 17$

B. $y = 4x - 7$

C. $y = 8x + 3$

D. $y = 8x - 15$

(g) **D**

(h) (5 points) What is $\frac{d}{dx} \arctan(x^2)$?

A. $\frac{2x}{1+x^2}$

B. $\frac{2x}{1+x^4}$

C. $\frac{1}{1+x^4}$

D. $\frac{2}{1+x^2} \arctan(x)$

(h) **B**

(i) (5 points) For which of the following f does $f'(x) = \ln(x) + 1$?

A. $f(x) = x \ln(x) + \frac{1}{x}$

B. $f(x) = x^2 \ln(x)$

C. $f(x) = x + \ln(x)$

D. $f(x) = x \ln(x) + 2$

(i) **D**

(j) (5 points) What is the value of $\frac{dy}{dx}$ at $(1, 1)$ of the curve determined implicitly by $x^3 + 2y^2 = 3$?

A. $-\frac{3}{4}$

B. $\frac{2}{3}$

C. -1

D. $\frac{1}{2}$

6. **A**