Problem 1 (12 points). Find the following.

(a) $\int (1-3x^2)(1+3x^2)\,dx$

$$= \int (1-9x^4)\,dx = \boxed{x - \frac{9x^5}{5} + C}$$

(b) $\int_1^2 \left(\frac{x}{2} - \frac{2}{x}\right) dx = \left(\frac{1}{2}\cdot\frac{x^2}{2} - 2\ln|x|\right)\Big|_1^2$

$$= \frac{4}{4} - 2\ln(2) - \left(\frac{1}{4} - 2\overset{0}{\cancel{\ln(1)}}\right)$$

$$= \boxed{\frac{3}{4} - 2\ln(2)}$$

(c) An equation for the tangent line to the graph of $y = x^{\sin(\pi x)}$ at the point $(1, 1)$.

The line will have pt-slope form $y - 1 = m(x-1)$,

where the slope $m = y'\big|_{x=1}$.

To compute y' we use logarithmic differentiation:

$$\ln(y) = \ln(x^{\sin(\pi x)}) = \sin(\pi x)\ln(x)$$

Hit both sides w/ $\frac{d}{dx}$:

$$\frac{y'}{y} = \pi\cos(\pi x)\ln(x) + \frac{\sin(\pi x)}{x}$$

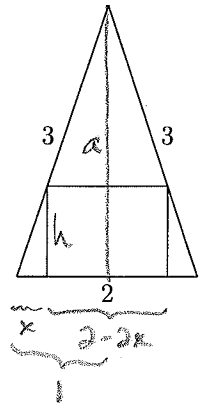
Evaluate at $(x,y) = (1,1)$:

$$\frac{y'|_{x=1}}{1} = \pi\cos(\pi)\overbrace{\cancel{\ln(1)}}^{0} + \frac{\overbrace{\cancel{\sin(\pi)}}^{0}}{1}$$

$$\Rightarrow m = y'\big|_{x=1} = 0$$

$\Rightarrow$ tan line has eqn $\boxed{y - 1 = 0, \text{ or } y = 1}$

Problem 2. [10 points] Find the dimensions of the rectangle of largest area that can be inscribed in an isosceles triangle of base 2 and other sides of length 3, where one side of the rectangle is along the base.



We want to maximize the area of the rectangle $A = h(2-2x)$

By the Pythagorean thm, the altitude of the triangle has length $a = \sqrt{3^2-1^2} = \sqrt{8}$
$= 2\sqrt{2}$

Similar triangles $\Longrightarrow$ $\dfrac{h}{x} = \dfrac{2\sqrt{2}}{1} \Longrightarrow h = 2\sqrt{2}x$

$$\Longrightarrow A(x) = 2\sqrt{2}x(2-2x) = 4\sqrt{2}x - 4\sqrt{2}x^2, \quad x \in [0,1]$$

To find the max value of A, we use the closed interval method.

Crit pts: set $0 = A'(x) = 4\sqrt{2} - 8\sqrt{2}x \Rightarrow 8\sqrt{2}x = 4\sqrt{2} \Rightarrow x = \frac{1}{2}$.

This is the only crit pt, which yields an area of
$A(\frac{1}{2}) = 2\sqrt{2}\cdot\frac{1}{2}(2 - 2\cdot\frac{1}{2}) = \sqrt{2}$

Endpts: $A(0) = 0$
$A(1) = 2\sqrt{2}(2-2) = 0$

So the biggest possible area occurs for $x = \frac{1}{2}$, which yields dimensions $2 - 2\frac{1}{2} = 1$ & $h = 2\sqrt{2}\cdot\frac{1}{2} = \sqrt{2}$.

Problem 3. [10 points] As a spherical ball of ice melts, its surface area is decreasing at a rate of 2 cm^2 per minute. What's the rate of change of the radius of the ball when the radius is 5 cm? (HINT: Be careful with your signs! Also, the area of a sphere of radius r is $4\pi r^2$.)

$$A = 4\pi r^2$$

$$\Rightarrow \frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

$$\Rightarrow -2 = 8\pi (5) \left.\frac{dr}{dt}\right|_{r=5}$$

$$\Rightarrow \left.\frac{dr}{dt}\right|_{r=5} = -\frac{1}{20\pi} \text{ cm/min.}$$

In other words, the radius is decreasing at a rate of $\frac{1}{20\pi}$ cm/min.

Problem 4. [10 points] Let

$$f(x) = \int_0^x e^{-t^2}\, dt.$$

Find all local maxima, local minima, and inflection points of f (if any), and where f is increasing and decreasing.

Since e^{-t^2} is everywhere cts, f is everywhere dble by FTC I.

Therefore all local maxes & mins occur where $f'(x)=0$.

By FTC I, $f'(x) = e^{-x^2}$, which is always positive!

So we conclude 1) f has no loc maxes or mins, &

2) f is everywhere increasing

It remains to find the inflection points, which occur where the concavity changes. We solve

$$0 = f''(x) = -2xe^{-x^2} \Rightarrow x = 0.$$

For negative x, $-2xe^{-x^2}$ is positive, & for positive x, $-2xe^{-x^2}$ is negative $\Rightarrow$ 0 is an inflection pt, & is moreover the only one.

Problem 5. [10 points]

(a) What is wrong with the following application of L'Hôpital's Rule? What should the limit really be?

$$\lim_{x\to 1} \frac{x^3+x-2}{x^2-3x+2} = \lim_{x\to 1} \frac{3x^2+1}{2x-3} = \lim_{x\to 1} \frac{6x}{2} = 3$$

The 1st limit is of type $\frac{0}{0}$, so the use of L'H is justified.

The 2nd lim is not of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so L'H can't be applied to it.

Instead $\lim_{x\to 1} \frac{x^3+x-2}{x^2-3x+2} \overset{L'H}{=} \lim_{x\to 1} \frac{3x^2+1}{2x-3} = \frac{4}{-1} = \boxed{-4}$ by direct substitution.

(b) Evaluate $\lim_{x\to 0} \frac{1}{x^2} \int_0^x \sin(t^2)\, dt$.

$= \lim_{x\to 0} \frac{\int_0^x \sin(t^2)dt}{x^2}$ type $\frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0} \frac{\sin(x^2)}{2x}$ (using FTCI in the numerator) type $\frac{0}{0}$ again!

$\overset{L'H}{=} \lim_{x\to 0} \frac{\cancel{2}x\cos(x^2)}{\cancel{2}} = 0 \cdot \cos(0) = \boxed{0}$