1 (30 pts.) Find the derivatives $f'(x)$ of the following functions $y = f(x)$.

a) $f(x) = \frac{x^2+\sin x}{\cos x}$

i) $f(x) = \frac{x^2}{\cos x} + \tan x \Rightarrow f' = \frac{2x\cos x - x^2(\cos x)'}{\cos^2 x} + \sec^2 x$

$= \frac{2x\cos x + x^2 \sin x}{\cos^2 x} + \sec^2 x$

$= 2x\sec x + x^2 \sec x \tan x + \sec^2 x$

ii) $f(x) = x^2 \sec x + \tan x$

$f'(x) = 2x\sec x + x^2 \sec x \tan x + \sec^2 x$

b) $f(x) = \ln(\sqrt{x^2+1})$

i) $y = \ln u \quad u = \sqrt{v} \quad v = x^2+1 \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} = \frac{1}{u}\frac{1}{2\sqrt{v}} 2x$

$= \frac{2x}{\sqrt{x^2+1} \cdot 2\sqrt{x^2+1}} = \frac{x}{x^2+1}$

ii) $f(x) = \ln(x^2+1)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2+1)$

$\Rightarrow y = \frac{1}{2}\ln u \quad u = x^2+1 \Rightarrow \frac{dy}{dx} = \frac{1}{2u} 2x = \frac{x}{x^2+1}$

c) $f(x) = (\sin x)^x \ (x \in (0, \pi))$

i) $y = f(x) = (\sin x)^x \Rightarrow \ln y = x \ln \sin x$

$\Rightarrow \frac{y'}{y} = \ln \sin x + x \frac{\cos x}{\sin x}$

$= \ln \sin x + x \cot x$

$\Rightarrow y' = (\sin x)^x [\ln \sin x + x \cot x]$

ii) $y = (\sin x)^x = e^{x \ln \sin x}$

$y' = e^{x\ln \sin x} \frac{d}{dx}(x \ln \sin x)$

$= e^{x \ln \sin x}\left(\ln \sin x + x\frac{\cos x}{\sin x}\right) = (\sin x)^x (\ln \sin x + x \cot x)$

2 (20 pts.) Evaluate the following limits.

a) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

$$\lim_{x\to\infty} \ln x = \lim_{x\to\infty} \sqrt{x} = \infty \qquad \infty/\infty$$

$$\Rightarrow \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

b) $\lim_{x\to\infty} (\frac{2x-1}{2x+2})^{x+1}$

$$\lim_{x\to\infty} \frac{2x-1}{2x+2} = \lim_{x\to\infty} \frac{2-1/x}{2+2/x} = \frac{2}{2} = 1$$

$$\lim_{x\to\infty} (x+1) = \infty$$

type 1^∞ $\qquad y = \left(\frac{2x-1}{2x+2}\right)^{x+1} \qquad \ln y = (x+1)\ln\left(\frac{2x-1}{2x+2}\right) = \frac{\ln(2x-1) - \ln(2x+2)}{1/(x+1)}$

type $0/0$

$$\therefore \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(2x-1) - \ln(2x+2)}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{\frac{1}{2x-1} - \frac{1}{2x+2}}{-\frac{1}{(x+1)^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3}{(2x-1)(2x+2)}}{-\frac{1}{(x+1)^2}} = \lim_{x\to\infty} -\frac{3(x+1)^2}{(2x-1)(2x+2)} = -\frac{3}{4}$$

$$\therefore \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^{-3/4}$$

3 (20 pts.)

The volume of a cylinder of radius r and height h is $\pi r^2 h$. The diagonal L of the cylinder satisfies Pythagoras relation $L^2 = h^2 + (2r)^2$.

Question. Of all cylinders of given diagonal length L, determine the height and radius of the one that has maximum volume. Argue your answer thoroughly.

volume $V = \pi r^2 h \qquad L^2 = h^2 + (2r)^2 \Rightarrow r^2 = \frac{L^2 - h^2}{4}$

range of h is from 0 ~~to~~ L. $\quad h \in [0, L]$

express V as function of h:

$$V = \pi r^2 h = \pi \frac{L^2 - h^2}{4} h = \frac{\pi}{4}(L^2 - h^2)h = \frac{\pi}{4}(L^2 h - h^3)$$

To find global maximum of V,

i) critical points of V: $V' = \frac{\pi}{4}(L^2 - 3h^2) = 0$

$$\Rightarrow h = \pm \frac{L}{\sqrt{3}}$$

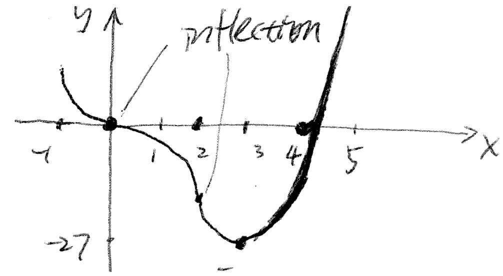
$$L \geq h \geq 0 \Rightarrow h = L/\sqrt{3}$$

$$V(L/\sqrt{3}) = \frac{\pi}{4}\left(L^2 - (L/\sqrt{3})^2\right) L/\sqrt{3} = \frac{\pi}{4} \cdot \frac{2}{3}L^2 \cdot \frac{L}{\sqrt{3}} = \frac{\pi L^3}{6\sqrt{3}}$$

ii) endpoints $V(0) = 0$, $V(L) = 0$

$\therefore$ maximum occurs when $h = L/\sqrt{3}$

radius $r = \frac{\sqrt{L^2 - h^2}}{2} = \frac{1}{2} \cdot \sqrt{\frac{2}{3}L^2} = \frac{L}{\sqrt{6}}$



4 (30 pts.) Consider the function $f(x) = x^4 - 4x^3$.

a) Determine the intervals where f is increasing or decreasing.

b) Determine the intervals where (the graph of) f is concave up(CU) or concave down(CD).

c) Determine local maxima and minima.

d) Determine the points where the graph of f intersects the x-axis.

e) Use the information from above to sketch the graph of f(x) over the interval $[-1, 5]$.

[To receive full credit you must exhibit the features from a)-d)]

$f(x) = x^4 - 4x^3 \Rightarrow f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

a) $f'(x) > 0 \Leftrightarrow x > 3 \Rightarrow$ increasing for $x > 3$

$f'(x) < 0 \Leftrightarrow x < 3$ and $x \neq 0 \Rightarrow$ decreasing for $x < 3$ (or $(-\infty, 0)$ & $(0,3)$)

b) $f'' > 0 \Leftrightarrow x > 2$ or $x < 0 \Rightarrow$ C.U.

$f'' < 0 \Leftrightarrow x \in (0,2) \Rightarrow$ C.D.

c) local max/min ~~occur when f~~ must be critical point s.t. $f'(x) = 0$.
(f differentiable ⇒)

$f'(x) = 0 \Leftrightarrow x = 0, 3$

for $x = 0$, f' does not change sign ⇒ not max/min

for $x = 3$, f' change from negative to positive ⇒ local minimum.

∴ No local maximum; local minimum is $f(3) = 3^4 - 4 \cdot 3^3 = -27$

d) $f(x) = 0 \Rightarrow x^3(x-4) = 0 \Rightarrow x = 0$ or 4

e) from a-d, we know: $[-1, 0)$ C.U. ↓ $(0,2)$ C.D. ↓ $(2,3)$ C.U. ↓ $[3,5]$ C.U. ↑

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