Practice Midterm II
110.108 Calculus I for Engineers Fall 2010
Solutions

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1. (a) False, $f'(x) = 0$ or $f'(x)$ does not exist.
(b) False, x^3 has a critical point at $x = 0$ but it is neither a maximum nor a minimum.
(c) True.
(d) True.
(e) False, $\int 2x + 1\ dx = x^2 + x + C$ (don't forget the constant!)

2. Use implicit differentiation. Doing so gives

$$3x^2y + x^3\frac{dy}{dx} = \left(e^y + xe^y\frac{dy}{dx}\right)\cos(e^yx) \Longrightarrow \frac{dy}{dx}\left(x^3 + xe^y\cos(e^yx)\right) = e^y\cos(e^yx) - 3x^2y$$

$$\Longrightarrow \frac{dy}{dx} = \frac{e^y\cos(e^yx) - 3x^2y}{x^3 + xe^y\cos(e^yx)}.$$

3. We're told that $A = \frac{r^2\theta}{2}$. If the area of the sector is changing with respect to time, then either the radius is changing over time or the angle is. But $r = 6$ is constant so it must be the angle, θ, that's changing. Therefore,

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt}.$$

In one minute, the minute hand will traverse 1/60th of the clock, or $\frac{360°}{60°} = 6°$ every minute. In radians, this is $\frac{\pi}{30}$ radians per minute. That is, $\frac{d\theta}{dt} = \frac{\pi}{30}$ rad/min. We're told that $r = 6$ inches so plugging these in,

$$\frac{dA}{dt} = \frac{(6\text{ in})^2}{2}\cdot\frac{\pi}{30}\text{ rad/min} = \frac{3\pi}{5}\text{ in}^2\text{/min}.$$

4. (a) The local linearity approximation is

$$f(x) \approx f(a) + f'(a)(x - a).$$

(b) In this case, $f(x) = \sin(x)$, $x = 179°$ and $a = 180°$. Then from the above equation,

$$\sin(179°) \approx \sin(180°) + \cos(180°)(179° - 180°).$$

The first term on the right is just 0, and $\cos(180°) = -1$. However, we have to convert $179° - 180° = -1°$ into radians, which is $-\frac{\pi}{180}$ radians. Hence

$$\sin(179°) \approx 0 - 1\cdot -\frac{\pi}{180} = \frac{\pi}{180}.$$

5. (a) We take the derivative of $f(x)$ and set it equal to zero to locate the critical points:

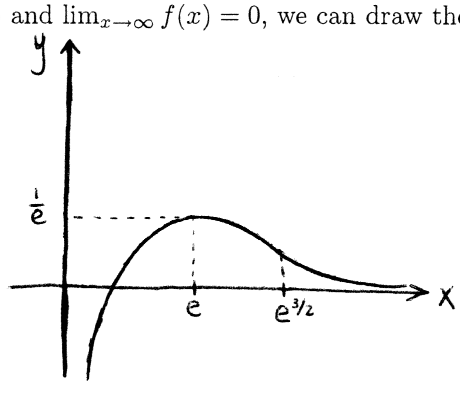
$$f'(x) = \left(\frac{\ln(x)}{x}\right)' = \frac{x\cdot\frac{1}{x} - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2} = 0 \Longrightarrow 1 - \ln(x) = 0 \Longrightarrow \ln(x) = 1.$$

It follows that $x = e$ is the only critical point because there are no points in $(0, \infty)$ where f' doesn't exist.

(b) At the critical point, $f(e) = \frac{\ln(e)}{e} = \frac{1}{e}$. Note that since $f'(1) = 1 > 0$, $f'(x) > 0$ for $x < e$ so $f(x)$ is increasing up to $x = e$ by the first derivative test. Similarly, $f'(10) = \frac{1-\ln(10)}{100} < 0$ so $f'(x) < 0$ for $x > e$, meaning $f(x)$ is decreasing after $x = e$. Therefore, $f(e) = \frac{1}{e}$ must be a *maximum.* Taking the second derivative,

$$f''(x) = \frac{2\ln(x) - 3}{x^3}\text{(check this!)}$$

Then the inflection points are when $f''(x) = 0 \Rightarrow 2\ln(x) - 3 = 0 \Rightarrow x = e^{3/2}$. When $x < e^{3/2}$, say at $x = 1$, $f''(1) = -3 < 0$ so $f(x)$ is concave *down.* When $x > e^{3/2}$, say at $x = e^2$, $f''(e^2) = \frac{1}{e^6} > 0$ so $f'(x)$ is concave *up.* Noting that $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = 0$, we can draw the graph below:



(c) From part (b), $f(e)$ is the maximum value of f so $f(e) > f(x)$ for all $x \neq e$. In particular, $f(e) > f(\pi)$.
(d) From part (c), $f(e) > f(\pi)$. Writing this out,

$$f(e) > f(\pi) \implies \frac{\ln(e)}{e} > \frac{\ln(\pi)}{\pi} \implies \frac{1}{e} > \frac{\ln(\pi)}{\pi}$$
$$\implies \pi > e\ln(\pi) \implies e^{\pi} > \pi^{e}.$$

6. Suppose that $a \neq b$ are both roots, so $f(a) = f(b) = 0$ where a, b are in $[0, 1]$. Then by Rolle's Theorem, there exists a point c in $(0, 1)$ such that $f'(c) = 3c^2 - 3 = 3(c^2 - 1) = 0$. But then $c = \pm 1$ and since neither 1 nor -1 is in $(0, 1)$ (note that Rolle's Theorem says there is a c in the *open* interval), we get a contradiction. Hence there can't be two roots. To see that both zero and one root are possible, when $m = -1$, there is one root (somewhere around 0.3) and when $m = 2$, there are none.

7. (a) The problem lies in the second application of L'Hôpital's rule. Recall that it can be used only when we have something of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and after we use it once, we get $\frac{4}{1} = 4$. So using it again is not only unnecessary, it's wrong. As we just saw, the actual limit is 4.
(b) We use L'Hôpital's rule because if we plug in 1, we get $\frac{0}{0}$:

$$\lim_{x \to 1} \frac{x\ln(x) - x + 1}{x^3 + x^2 - 5x + 3} = \lim_{x \to 1} \frac{\ln(x)}{3x^2 + 2x - 5}$$

by the product rule applied to the numerator, and the power rule to the denominator. Plugging in 1, we still have something of the form $\frac{0}{0}$ so we go again:

$$\lim_{x \to 1} \frac{\ln(x)}{3x^2 + 2x - 5} = \lim_{x \to 1} \frac{1}{x} \cdot \frac{1}{6x + 2} = \frac{1}{8}.$$