Math 201 Midterm Exam 1 — Mar. 10, 2017

Name: ______________________________

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Section Time (Circle one): T 3:00-3:50 T 4:30-5:20 Th 1:30-2:20 Th 3:00-3:50

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T 1:30-2:20 T 3:00-3:50 Th 3:00-3:50 Th 4:30-5:20 T 4:30-5:20

- Complete the following problems. In order to receive full credit, please make sure to *justify your answers.* You are free to use results from class or the course textbook as long as you clearly state what you are citing.
- **You have 50 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted (nor are they needed). If you finish early, you must hand your exam paper to a proctor.
- Please check that your copy of this exam contains 8 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

The following boxes are strictly for grading purposes. Please do not mark.

Question	Points	Score
1	15	
2	25	
3	20	
4	15	
5	25	
Total	100	

1. (15 points) Use Gauss-Jordan elimination to compute rref(A) for $A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix} \begin{array}{l} \\ \\ -3\mathrm{I} \\ \\ \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 4 & -12 & -4 \\ 0 & -1 & 3 & 4 \end{bmatrix} \begin{array}{l} \\ \\ -4\mathrm{II} \\ +\mathrm{II} \end{array} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{array}{l} \\ \\ \\ \div 3 \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} -4\mathrm{IV} \\ +\mathrm{IV} \\ \\ \\ \end{array} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. Consider the matrix $A = \begin{bmatrix} 0 & 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) (5 points) Find *one* solution to $A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$.

The general solution of the system is

$$x_2 = x_3 - 2x_4 - x_6 + 1$$
$$x_5 = -3x_6 - 2.$$

Letting $x_1 = x_3 = x_4 = x_6 = 0$, one solution of the system can be $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -2 \\ 0 \end{bmatrix}$.

(b) (5 points) Find a vector $\vec{y}$ so that the system $A\vec{x} = \vec{y}$ has *no* solutions.

Let $\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; then the last row of the system $A\vec{x} = \vec{y}$ gives "$0 = 1$", which means that the system is inconsistent and has no solution. So $\vec{y}$ can be $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) (5 points) Determine $\text{null}(A) := \dim\ker(A)$.

Note that A is already in rref and has 2 leading 1's, so $\text{rank}(A) = 2$. By the rank-nullity theorem, $\text{null}(A) = \dim\ker(A) = 6 - \text{rank}(A) = 4$.

(d) (10 points) Recall, $A = \begin{bmatrix} 0 & 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Determine a basis of $\ker(A)$.

The solution of the system $A\vec{x} = \vec{0}$ is

$$x_2 = x_3 - 2x_4 - x_6$$
$$x_5 = -3x_6 \quad , \quad \text{i.e.}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} s \\ t-2u-w \\ t \\ u \\ -3w \\ w \end{bmatrix}$$

$$= \begin{bmatrix} s \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2u \\ 0 \\ u \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -w \\ 0 \\ 0 \\ -3w \\ w \end{bmatrix}$$

$$= s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

Therefore, a basis of the kernel is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

- Note that x_1 does not appear in the system $A\vec{x} = \vec{0}$, which means that there is no restriction on x_1. So x_1 can be any number – it is also a free variable! Don't forget this when forming the solution of the system.

3. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$. Let $A^2 = A \cdot A$, $A^3 = A \cdot A^2$ and I_3 denote the 3×3 identity matrix.

(a) (10 points) Verify that $A^3 + A = 2I_3$.

$$A^2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & 1 \\ -1 & -1 & 0 \end{bmatrix};$$

$$A^3 = A\,A^2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix};$$

thus, $$A^3 + A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= 2 I_3.$$

(b) (10 points) Explain why A is invertible and determine its inverse.

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ -\mathrm{I} \end{matrix} \rightarrow \begin{bmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \times(-1) \\ \\ \end{matrix} \rightarrow$$

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ +\mathrm{II} \end{matrix} \rightarrow \begin{bmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{bmatrix} \begin{matrix} \\ \\ \div 2 \end{matrix} \rightarrow$$

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{matrix} +\mathrm{III} \\ -\mathrm{III} \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The left half of the last matrix is I_3. This means that the system $A\vec{x} = \vec{y}$ has a unique solution for any $\vec{y}$ in $\mathbb{R}^3$. Therefore, A is invertible, and

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

- You can also derive the invertibility of A and compute A^{-1} from the identity in (a):

$$\begin{aligned}
&A^3 + A = 2I_3 \Rightarrow \\
&A^3 + AI_3 = 2I_3 \Rightarrow \\
&A(A^2 + I_3) = 2I_3 \Rightarrow \\
&A\left[\tfrac{1}{2}(A^2 + I_3)\right] = I_3 \Rightarrow \\
&A \text{ is invertible and } A^{-1} = \tfrac{1}{2}(A^2 + I_3).
\end{aligned}$$

4. (15 points) Let $T : \mathbb{R}^m \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^n$ be linear transformations. Define $H : \mathbb{R}^m \to \mathbb{R}^n$ by

$$H(\vec{x}) = S(T(\vec{x}) + 2\vec{x}).$$

Verify that H is a linear transform and determine $[H]$, the standard matrix of H, in terms of $[T]$ and $[S]$, the standard matrices of T and S.

For any $\vec{x}$, $\vec{y}$ in $\mathbb{R}^m$ and any scalar k,

$$\begin{aligned} H(\vec{x}+\vec{y}) &= S(T(\vec{x}+\vec{y}) + 2(\vec{x}+\vec{y})) \\ &= S(T(\vec{x}) + T(\vec{y}) + 2\vec{x} + 2\vec{y}) \quad \text{since } T \text{ is a linear transformation} \\ &= S((T(\vec{x}) + 2\vec{x}) + (T(\vec{y}) + 2\vec{y})) \\ &= S(T(\vec{x}) + 2\vec{x}) + S(T(\vec{y}) + 2\vec{y}) \quad \text{since } S \text{ is a linear transformatio} \\ &= H(\vec{x}) + H(\vec{y}) ; \end{aligned}$$

$$\begin{aligned} H(k\vec{x}) &= S(T(k\vec{x}) + 2k\vec{x}) \\ &= S(kT(\vec{x}) + 2k\vec{x}) \quad \text{since } T \text{ is a linear transformation} \\ &= S(k(T(\vec{x}) + 2\vec{x})) \\ &= k\,S(T(\vec{x}) + 2\vec{x}) \quad \text{since } S \text{ is a linear transformation} \\ &= kH(\vec{x}). \end{aligned}$$

Therefore, H is a linear transformation.

By linearity of S, $H(\vec{x}) = S(T(\vec{x}) + 2\vec{x})$

$$\begin{aligned} &= S(T(\vec{x})) + S(2\vec{x}) \\ &= S(T(\vec{x})) + 2S(\vec{x}) ; \end{aligned}$$

this shows $[H] = [S][T] + 2[S]$.

5. (a) (5 points) State what it means for $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$ to be linearly independent.

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^n$ are linearly independent if they have only the trivial linear relation, i.e. if the system

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$ has only the zero solution $c_1 = c_2 = c_3 = 0$.

- Alternative definitions: (1) $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent if none of the vectors is a linear combination of the other vectors.

  (2) $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent if none of the vectors is a linear combination of the preceding vectors.

(b) (10 points) Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$ are linearly independent. Show that

$$\vec{w}_1 = \vec{v}_1 + \vec{v}_2, \quad \vec{w}_2 = \vec{v}_2 + \vec{v}_3, \quad \vec{w}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$$

are linearly independent.

Form the system $c_1 \vec{w}_1 + c_2 \vec{w}_2 + c_3 \vec{w}_3 = \vec{0}$. We want to show that the only solution of this system is $c_1 = c_2 = c_3 = 0$.

Substituting for $\vec{w}_1$, $\vec{w}_2$ and $\vec{w}_3$, we have

$$c_1 (\vec{v}_1 + \vec{v}_2) + c_2 (\vec{v}_2 + \vec{v}_3) + c_3 (\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}.$$

Grouping terms gives

$$(c_1 + c_3) \vec{v}_1 + (c_1 + c_2 + c_3) \vec{v}_2 + (c_2 + c_3) \vec{v}_3 = \vec{0}.$$

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, we must have

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

The solution of this system is easily seen to be $c_1 = c_2 = c_3 = 0$. Therefore, $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are linearly independent.

(c) (10 points) Suppose that $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$. Show that the vectors

$$\vec{w}_1 = \vec{v}_1 - \vec{v}_2 - \vec{v}_3, \quad \vec{w}_2 = 2\vec{v}_1 - 3\vec{v}_2 - \vec{v}_3, \quad \vec{w}_3 = \vec{v}_3 - \vec{v}_2$$

are *not* linearly independent.

By inspection, $2\vec{w}_1 - \vec{w}_2 + \vec{w}_3 = \vec{0}$.

($2\vec{w}_1 - \vec{w}_2 + \vec{w}_3 = 2(\vec{v}_1 - \vec{v}_2 - \vec{v}_3) - (2\vec{v}_1 - 3\vec{v}_1 - \vec{v}_3) + (\vec{v}_3 - \vec{v}_2) = \vec{0}$.

To come up with this linear relation, just try to make the coefficient of $\vec{v}_1$ 0 first, and then make the coefficient of $\vec{v}_2$ 0, and finally verify that the coefficient of $\vec{v}_3$ is 0. You may also set $c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{w}_3 = \vec{0}$

$\Rightarrow c_1(\vec{v}_1 - \vec{v}_2 - \vec{v}_3) + c_2(2\vec{v}_1 - 3\vec{v}_2 - \vec{v}_3) + c_3(\vec{v}_3 - \vec{v}_2) = \vec{0}$

$\Rightarrow (c_1 + 2c_2)\vec{v}_1 + (-c_1 - 3c_2 - c_3)\vec{v}_2 + (-c_1 - c_2 + c_3)\vec{v}_3 = \vec{0}$,

and then set

$$c_1 + c_2 = 0$$
$$-c_1 - 3c_2 - c_3 = 0$$
$$-c_1 - c_2 + c_3 = 0$$

and find a nonzero solution for c_1, c_2, c_3.)

Therefore $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are not linearly independent.