Homework 01 Solution Key

Problem 01)

(a)
$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & -3 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

(b)
$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & -3 \\ 0 & 1 & -1 & 9 \end{array}\right]$$

Problem 02)

(a)
$$\begin{aligned} 0x_1 + x_2 + 2x_3 &= -1 \\ 2x_1 + 0x_2 + 0x_3 &= 0 \\ x_1 - x_2 + 9x_3 &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 0x_2 - x_3 + 2x_4 &= 0 \\ 2x_1 + 0x_2 + 0x_3 + 7x_4 &= 1 \end{aligned}$$

Problem 03)

(a) not in rref; the leading 1 in row 2 is not the only non-zero entry in its column.

(b) yes, in rref

(c) not in rref; the leading 1 in row 1 is to the right of the leading 1 in row 2.

(d) not in rref; row 2 doesn't have a leading 1.

Problem 04)

(a) $$\begin{bmatrix} 2 & 6 & -2 \\ 3 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{II}-\text{III}} \begin{bmatrix} 2 & 6 & -2 \\ 3 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\frac{1}{3}\times \text{II}} \begin{bmatrix} 2 & 6 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{I}-2\text{II}} \begin{bmatrix} 0 & 6 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{I}+2\text{III}} \begin{bmatrix} 0 & 6 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{6}\times \text{I}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Rearrange}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 3 & -3 \\ 1 & 1 \\ 0 & 0 \\ 2 & 4 \end{bmatrix} \xrightarrow{\text{I}-3\text{II}} \begin{bmatrix} 0 & -6 \\ 1 & 1 \\ 0 & 0 \\ 2 & 4 \end{bmatrix} \xrightarrow{-\frac{1}{6}\times \text{I}} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 2 & 4 \end{bmatrix} \xrightarrow{\text{II}-\text{I}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 2 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{IV}-2\text{II}-4\text{I}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{Rearrange}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 05)

(a) $$\begin{bmatrix} 2 & -8 & 2 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{II}-\text{I}} \begin{bmatrix} 2 & -8 & 2 \\ 0 & 9 & -1 \end{bmatrix} \xrightarrow{\frac{-1}{9}\times \text{II}} \begin{bmatrix} 2 & -8 & 2 \\ 0 & 1 & -1/9 \end{bmatrix} \xrightarrow{\text{I}+8\text{II}} \begin{bmatrix} 2 & 0 & 10/9 \\ 0 & 1 & -1/9 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}\times \text{I}} \begin{bmatrix} 1 & 0 & 5/9 \\ 0 & 1 & -1/9 \end{bmatrix} \Longrightarrow \begin{aligned} x &= 5/9 \\ y &= -1/9 \end{aligned}$$

(b) $$\begin{bmatrix} 1 & -2 & -1 & 2 \\ 1 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{\text{I}-\text{II}} \begin{bmatrix} 0 & -2 & 2 & 0 \\ 1 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{2}\text{I}} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & -3 & 2 \end{bmatrix}$$

$$\Longrightarrow \begin{aligned} x &= 2 + 3z \\ y &= z \end{aligned}$$

Problem 05 continued)

(c) $\begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & -1 & 3 & 4 & 0 \end{bmatrix} \xrightarrow{\mathrm{III}+\mathrm{II}} \begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \xrightarrow{\frac{1}{6}\times\mathrm{III}} \begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

$\xrightarrow{\mathrm{II}+\mathrm{III}} \begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & 0 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\mathrm{I}-4\times\mathrm{III}} \begin{bmatrix} 1 & 0 & 2 & 0 & -16 \\ 0 & 1 & -3 & 0 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = -16 - 2x_3 \\ x_2 = 8 + 3x_3 \\ x_4 = 2 \end{array}$

Problem 06)

(a) $\left[\begin{array}{cc|c} 1 & -2 & k \\ 3 & 1 & 1 \end{array}\right] \xrightarrow{\mathrm{III}-3\mathrm{I}} \left[\begin{array}{cc|c} 1 & -2 & k \\ 0 & 7 & 1-3k \end{array}\right] \xrightarrow{\frac{1}{7}\times\mathrm{II}} \left[\begin{array}{cc|c} 1 & -2 & k \\ 0 & 1 & \frac{1-3k}{7} \end{array}\right] \xrightarrow{\mathrm{I}+2\mathrm{II}} \left[\begin{array}{cc|c} 1 & 0 & k + \frac{2(1-3k)}{7} \\ 0 & 1 & \frac{1-3k}{7} \end{array}\right]$

$\Rightarrow \begin{array}{l} x = \frac{2}{7} + \frac{1}{7}k \\ y = \frac{1}{7} - \frac{3}{7}k \end{array} \Rightarrow$ a particular value of $k \in \mathbb{R}$ gives exactly one solution.

(b) $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 0 & -1 & k \\ 4 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\mathrm{I}-\mathrm{II}} \left[\begin{array}{ccc|c} 0 & 1 & 3 & 2-k \\ 1 & 0 & -1 & k \\ 4 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\mathrm{III}-4\mathrm{II}} \left[\begin{array}{ccc|c} 0 & 1 & 3 & 2-k \\ 1 & 0 & -1 & k \\ 0 & 1 & 3 & -4k \end{array}\right]$

$\xrightarrow{\mathrm{I}-\mathrm{III}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 2-k+4k \\ 1 & 0 & -1 & k \\ 0 & 1 & 3 & -4k \end{array}\right] \Rightarrow$ if $k \neq -2/3$, system is inconsistent $\Rightarrow$ no sol'ns

if $k = -2/3$, ∞-many solutions

Problem 07)

(a)

$$\begin{bmatrix} 2 & 0 & 4 \\ -1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}\mathrm{III}} \begin{bmatrix} 2 & 0 & 4 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}\mathrm{I}} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\mathrm{II}-\mathrm{III}} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\mathrm{I}+\mathrm{II}} \begin{bmatrix} 0 & 0 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}\mathrm{I}} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & \cancel{1} \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\mathrm{II}-\mathrm{I}} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{-1\times\mathrm{II}} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Rearrange}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \operatorname{rank}(A) = 3.$$

(b)

$$\begin{bmatrix} 2 & 2 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\mathrm{II}+\mathrm{III}} \begin{bmatrix} 2 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\mathrm{I}-2\mathrm{II}} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\mathrm{I}-2\mathrm{III}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{Rearrange}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Problem 08)

(a) no solutions or ∞-many $\Rightarrow$ not enough info

(b) inconsistent $\Rightarrow$ no solution

(c) no solution / one solution $\Rightarrow$ not enough info

(d) $\Rightarrow$ single solution

Problem 09).

(a) $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ -3 \\ -6 \end{bmatrix}$. Note that we can write

$$\frac{1}{2}\vec{v}_1 - \frac{1}{6}\vec{v}_2 = \frac{1}{2}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 0 \\ -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1/2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So $\vec{b}$ can be written as a linear combination of $\vec{v}_1$ & $\vec{v}_2$.

(b) $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Set the vectors as columns of an augmented matrix.

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} \xrightarrow{I + III} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} \Longrightarrow$$

Since the matrix is inconsistent, $\vec{b}$ can't be written as a linear combination of $\vec{v}_1$ & $\vec{v}_2$.

Problem 10)

(a) Set $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix}$. We have

$$b_1 = v_{11}x_1 + v_{21}x_2 + v_{31}x_3 = v_{11}x_1' + v_{31}x_2' + v_{21}x_3'$$
$$b_2 = v_{12}x_1 + v_{22}x_2 + v_{32}x_3 = v_{12}x_1' + v_{32}x_2' + v_{22}x_3'$$
$$b_3 = v_{13}x_1 + v_{23}x_3 + v_{33}x_3 = v_{13}x_1' + v_{33}x_2' + v_{23}x_3'$$

Matching up coefficients gives us $x_2' = x_3$

$\Longrightarrow \vec{x}' = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$ is a sol'n.

(b) Same set-up as above:

$$b_1 = 2v_{11}x_1'' + v_{21}x_2'' + v_{31}x_3'' = v_{11}x_1 + v_{21}x_2 + v_{31}x_3$$
$$b_2 = 2v_{12}x_1'' + v_{22}x_2'' + v_{32}x_3'' = v_{12}x_1 + v_{22}x_2 + v_{32}x_3$$
$$b_3 = 2v_{13}x_1'' + v_{23}x_2'' + v_{33}x_3'' = v_{13}x_1 + v_{23}x_2 + v_{33}x_3$$

Matching up coefficients gives $2x_1'' = x_1$

$\Longrightarrow \vec{x}'' = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ is a solution.