Solution of Homework 2

1. Since the matrix is of rank 2, it has two leading 1's. Recall that in a rref, a leading 1 is the first nonzero entry in a row; if a column contains a leading 1, then all other entries in that column are 0; as you move down the matrix, the leading 1's move to the right. Based on these observations, it's easy to write out all 3×3 matrices in rref which are of rank 2:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$ where a, b can be any number in $\mathbb{R}$.

2. (a) Since $A\begin{bmatrix}1\\2\end{bmatrix} = A\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ and linear transformations are linear, it follows that $A\left(\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}-1\\1\end{bmatrix}\right) = A\begin{bmatrix}1\\2\end{bmatrix} - A\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. Therefore, we can choose $\vec{x}$ to be

$$\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}.$$

(b) We need to find A explicitly. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ gives $\begin{array}{ll} a+2b=1 & ① \\ c+2d=0 & ② \end{array}$.

Similarly, $A\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ gives $\begin{array}{ll} -a+b=1 & ③ \\ -c+d=0 & ④ \end{array}$.

① & ③ $\Rightarrow a = -\frac{1}{3}$, $b = \frac{2}{3}$; ② & ④ $\Rightarrow c = 0$, $d = 0$.

Therefore, $A = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$.

Solving the system $A\vec{x} = \begin{bmatrix}0\\0\end{bmatrix}$, we immediately get $\vec{x} = \begin{bmatrix}2t\\t\end{bmatrix}$, where $t\in\mathbb{R}$ is arbitrary.

3. (a) Every component function of T is linear in x_1, x_2, x_3, so T is a linear transformation. The matrix of T is

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(b) T is not a linear transformation because of the constant term 1 in the second component function of T.

4. Idea: Convert each statement to an equivalent statement concerning the number of solutions of a linear system.

(a) Saying that $T(\vec{x}) \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for all $\vec{x}$ in $\mathbb{R}^3$ amounts to saying that the linear system $A\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has no solution, where A is the 2×3 matrix of T.

If the system is inconsistent (e.g. when $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$), then it has no solution.

If the system is consistent (e.g. when $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$), then, because it has two equations (so at most two leading variables) but a total of three variables, it will have infinitely many solutions.

Therefore, the statement holds true for some T.

(b) Saying that $T(\vec{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for some nonzero $\vec{x} \in \mathbb{R}^3$ amounts to saying that the linear system $A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has a nonzero solution.

Observe that the system $A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is always consistent: $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is always a solution. From another perspective, no matter what row operations we perform on the augmented matrix, its last column will remain a column of zeros, there can not be a nonzero entry in it. Thus, by the reason given at the end of (a), $A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ must have infinitely many solutions, and of course, have a nonzero solution, whatever A is.

The statement is true for all T.

(c) Saying that $T(\vec{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ only when $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ amounts to saying that the system $A\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has a unique solution $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

However, if $A\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has a solution, then it is consistent, and by the reason given at the end of (a), it will have infinitely many solutions. That it has a unique solution is impossible.

The statement holds true for no T.

(d) Recall that if $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, and if $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to$ ith, $i = 1, 2, \cdots, m$ are the standard vectors,

then $T\vec{e}_i$ is nothing but the ith column of the matrix A of T:

$A = \begin{bmatrix} | & | & & | \\ T\vec{e}_1 & T\vec{e}_2 & \cdots & T\vec{e}_m \\ | & | & & | \end{bmatrix}$. Thus, $T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and

$T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ tell us that the second column of the matrix A of T is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and the third column is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. But the first column is not determined: $A = \begin{bmatrix} - & 1 & 1 \\ - & -1 & 0 \end{bmatrix}$.

The statement is true for some T.

5 (a) We seek to express $\begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} = a\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ for some scalars a and b, and then apply the linearity of T: $T(a\vec{v} + b\vec{w}) = aT(\vec{v}) + bT(\vec{w})$. Solving the system $\begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} = a\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$,

we obtain $a=-1$, $b=1$. Therefore, $T\begin{bmatrix}-1\\3\\-3\end{bmatrix} = T\left(-\begin{bmatrix}2\\-2\\1\end{bmatrix}+\begin{bmatrix}1\\1\\2\end{bmatrix}\right) =$

$-T\begin{bmatrix}2\\-2\\1\end{bmatrix} + T\begin{bmatrix}1\\1\\2\end{bmatrix} = -\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}.$

(b) We try to solve $\begin{bmatrix}1\\0\\1\end{bmatrix} = a\begin{bmatrix}2\\-2\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\-2\end{bmatrix}$ but end up finding that the system has no solution. Since all we know is $T\begin{bmatrix}2\\-2\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\\-2\end{bmatrix}$ and the linearity of T, all we can determine is $T\left(a\begin{bmatrix}2\\-2\\1\end{bmatrix}+b\begin{bmatrix}1\\1\\-2\end{bmatrix}\right)$ Therefore, it's impossible to determine $T\begin{bmatrix}1\\0\\1\end{bmatrix}$ based on the information we have.

6. (a) Suppose that the matrix A of T is $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Then

$$T\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \begin{array}{ll} a-b=1 & (1)\\ c-d=0 & (2);\end{array}$$

$$T\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \Rightarrow \begin{array}{ll} 2b=2 & (3)\\ 2d=0 & (4)\end{array}$$

(3) and (4) yield $b=1$, $d=0$. plugging these into (1) and (2), we immediately obtain $a=2$, $c=0$. Therefore, the linear transformation that satisfies the requirement of the problem is represented by the matrix $A=\begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix}$.

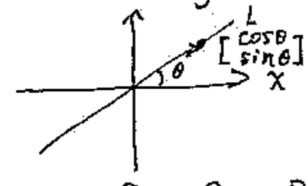
(b) Note that $\begin{bmatrix}4\\-2\end{bmatrix} = -2\begin{bmatrix}-2\\1\end{bmatrix}$, but $T\begin{bmatrix}4\\-2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \neq -2T\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}0\\-2\end{bmatrix}$. This violates the equivalent definition of linear transformation. So there is no linear transformation that can fulfill the requirement of the problem.

7. (a) $R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, $R_{-\theta} = \begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix}$

$= \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}.$

Since the direction of the x-axis is represent by the unit vector $\begin{bmatrix}1\\0\end{bmatrix}$, the matrix of the orthogonal projection onto the x-axis is $P_1 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$. In this problem, I will use the same notation for a linear transformation and its matrix.

Since the line L through the origin makes angle θ with the x-axis, its direction can be represented by the unit vector $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$; see picture below. Hence, the matrix of the orthogonal projection onto L is $P_L = \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$.



The problem wants us to verify the relation $P_L = R_\theta \circ P_1 \circ R_{-\theta}$.

Recalling that the composition of linear transformations corresponds to matrix mulplication, we need only verify $P_L = R_\theta P_1 R_{-\theta}$, with P_L, R_θ, P_1, $R_{-\theta}$ understood to be matrices. We thus compute

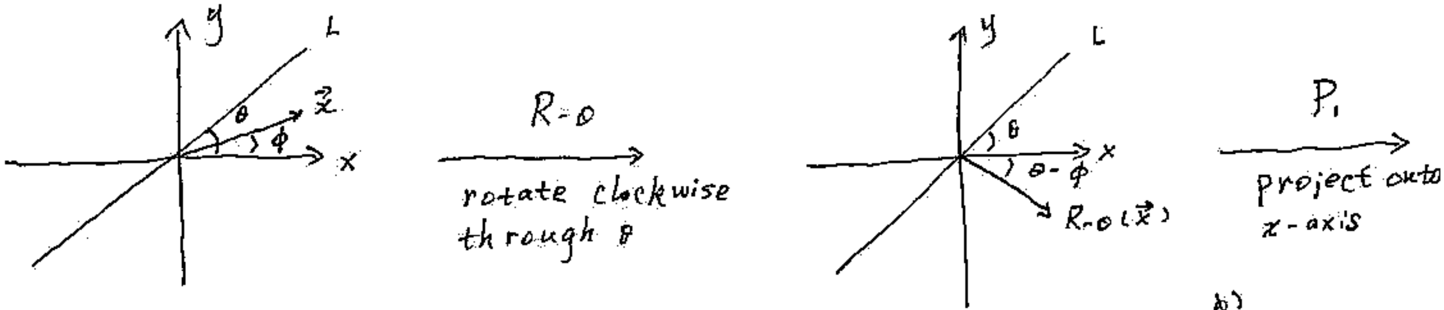
$$R_\theta P_1 R_{-\theta} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

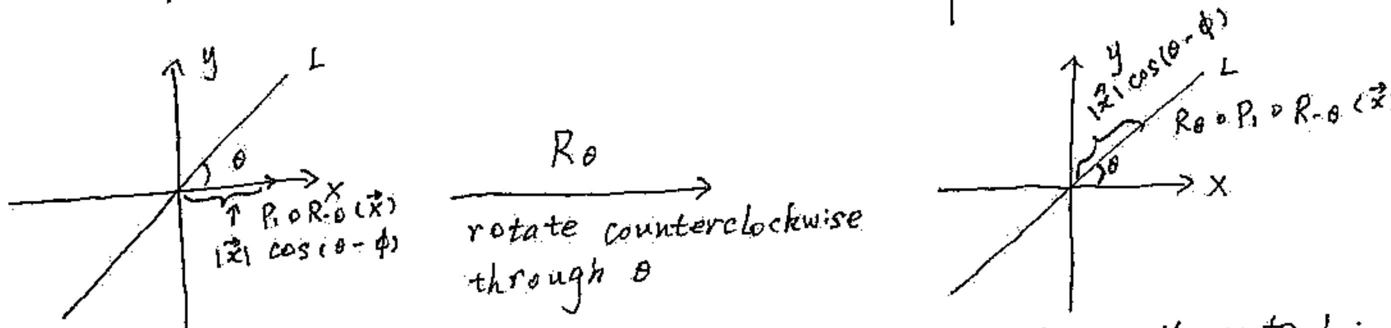
$$= \begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix},$$ which is indeed P_L.

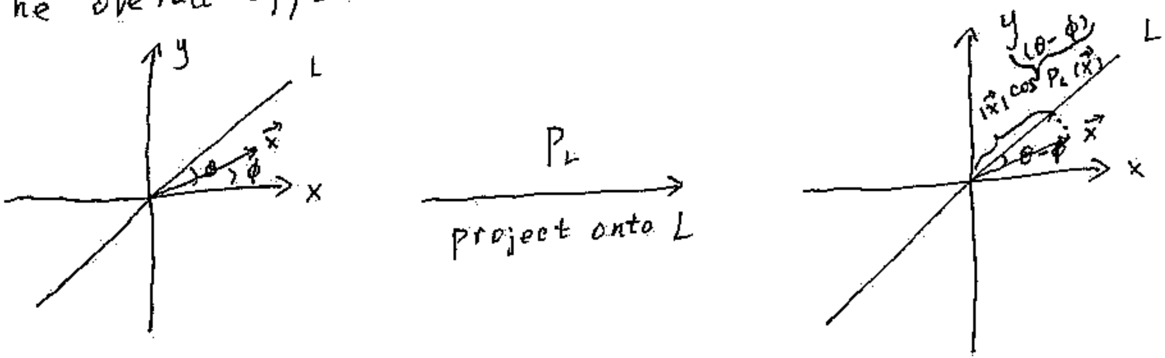
We therefore verified the relation $P_L = R_\theta \circ P_1 \circ R_{-\theta}$ algebraically.

(b) To intepret the relation $P_L = R_\theta \circ P_1 \circ R_{-\theta}$ geometrically, note that in the composition $R_\theta \circ P_1 \circ R_{-\theta}(\vec{x})$, $R_{-\theta}$ acts first, then P_1 and finally R_θ. (This is in accordance with the composition of functions; think about $f \circ g \circ h(x) = f(g(h(x)))$. The order matters, just as the order of matrix mulplication matters. Don't invert it.) The relation $P_L = R_\theta \circ P_1 \circ R_{-\theta}$ thus says that if we first rotate a vector $\vec{x}$ <u>clockwise</u> through angle θ, then project the result orthogonally onto the x-axis, and at last rotate counterclockwise through angle θ, the overall effect is the same as projecting the vector $\vec{x}$ onto the line L through the origin that makes angle θ with the x-axis. This can be seen from the following string of pictures.





The overall effect is the same as projecting orthogonally onto L:



8. Suppose $B = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} =$

$\begin{bmatrix} x_1 - x_3 & x_2 - x_4 \\ x_3 & x_4 \end{bmatrix}$. $BA = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & -x_1 + x_2 \\ x_3 & -x_3 + x_4 \end{bmatrix}$.

Equating the above two matrix products immediately yields

$$\begin{aligned} x_1 &= x_1 - x_3 \\ -x_1 + x_2 &= x_2 - x_4 \\ x_3 &= x_3 \\ -x_3 + x_4 &= x_4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_3 &= 0 \\ x_1 &= x_4. \end{aligned}$$

Therefore, the 2×2 matrices that commute with A are $B = \begin{bmatrix} x_1 & x_2 \\ 0 & x_1 \end{bmatrix}$, where x_1, x_2 can be any real numbers.

9. (a) $A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$. Therefore, $A^2 - 2A =$

$$\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

(b) $C^2 = C \cdot C = B^{-1}ABB^{-1}AB = B^{-1}A(BB^{-1})AB = B^{-1}AI_2AB = B^{-1}AAB = B^{-1}(AA)B = B^{-1}A^2B$, by associativity of matrix multiplication.

Therefore, $C^2 - 2C = B^{-1}A^2B - 2B^{-1}AB = B^{-1}A^2B - B^{-1}(2A)B = B^{-1}(A^2 - 2A)B = B^{-1}I_2B = B^{-1}B = I_2$, by distributivity of matrix multiplication and the fact that $A^2 - 2A = I_2$.

10. (a) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{matrix} \\ -2\mathrm{I} \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{matrix} -\mathrm{II} \\ \\ \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix}$.

Since the left half of the last matrix is I_2, A is invertible and $A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ -\mathrm{I} \\ -2\mathrm{I} \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ -\mathrm{II} \end{matrix} \longrightarrow$

$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{bmatrix} \begin{matrix} \\ \\ \times(-1) \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix} \begin{matrix} +\mathrm{III} \\ \\ \\ \end{matrix} \longrightarrow$

$\begin{bmatrix} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}$.

Since the left half of the last matrix in the above string is I_3, A is invertible, and $A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

Book problems

2.1, 4 Just think about how the product $A\vec{x}$ of a matrix A with a column vector $\vec{x}$ is defined: the ith component of $A\vec{x}$ is obtained by multiplying entries in the ith row of A with corresponding entries in $\vec{x}$ and then taking sum. Then it's easy to write out the matrix of the linear transformation:

$$\begin{bmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{bmatrix}.$$

2.1, 6 Every component function of T is a linear function of x_1 and x_2.

Yes. $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

2.1, 44 Same reason as in 2.1, 6. Yes. $\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$.

2.2, 8 The matrix is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ where $a^2+b^2=1$ ($a=0$, $b=-1$), so it gives a reflection about a line through the origin. To find the line of reflection, suppose its direction is represented by the unit vector $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Then the matrix of the reflection about this line is $\begin{bmatrix} 2u_1^2-1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2-1 \end{bmatrix}$ ($S = 2P - I_2$, Page 64 of the textbook). Equating this matrix with $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, we immediately solve $u_1 = \frac{\sqrt{2}}{2}$, $u_2 = -\frac{\sqrt{2}}{2}$ (or $u_1 = -\frac{\sqrt{2}}{2}$, $u_2 = \frac{\sqrt{2}}{2}$). So the matrix gives a reflection about the line $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$ through the origin (i.e. the line $y = -x$).

2.2 10 Formula on Page 63 of our textbook. $\begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$.

2.2,30 A can be chosen to be the matrix representing the orthogonal projection onto the line $\begin{bmatrix}1\\2\end{bmatrix}$ through the origin. Then $A\vec{x}$ will be parallel to $\begin{bmatrix}1\\2\end{bmatrix}$ for any $\vec{x} \in \mathbb{R}^2$.

$$A = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}.$$

2.3,12 $\begin{bmatrix} 0 & 1 \end{bmatrix}$

2.3,24 Suppose that ~~the matrix~~ B is a matrix that commutes with A: $AB = BA$. In order that the products on both sides of the equality be defined, B must be of size 3×3. Suppose

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}. \qquad AB = \begin{bmatrix} 2b_{11} & 2b_{12} & 2b_{13} \\ 2b_{21} & 2b_{22} & 2b_{33} \\ 3b_{31} & 3b_{32} & 3b_{33} \end{bmatrix},$$

$$BA = \begin{bmatrix} 2b_{11} & 2b_{12} & 3b_{13} \\ 2b_{21} & 2b_{22} & 3b_{23} \\ 2b_{31} & 2b_{32} & 3b_{33} \end{bmatrix}.$$

Equating these two products, we immediately get

$$\begin{aligned} 2b_{13} &= 3b_{13} \\ 2b_{23} &= 3b_{23} \\ 2b_{31} &= 3b_{31} \\ 2b_{32} &= 3b_{32} \end{aligned} \quad \Rightarrow \quad b_{13} = b_{23} = b_{31} = b_{32} = 0.$$

Therefore, the matrices that commute with A are

$$\begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$$ where $b_{11}, b_{12}, b_{21}, b_{22}, b_{33}$ can be any real number.

2.3,44. Consider geometrical transformations. If A is the matrix of rotation through angle $\frac{\pi}{2}$, then A^2 represents rotation through angle π, which is not the identity map, but A^4 is the identity map (rotation through angle 2π). Therefore, such an A satisfies the requirement of the problem. A can be chosen to be

$$\begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

2.4, 30 We compute

$$\left[\begin{array}{ccc|ccc} 0 & 1 & b & 1 & 0 & 0 \\ -1 & 0 & c & 0 & 1 & 0 \\ -b & -c & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \\ +c\mathrm{I} \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & b & 1 & 0 & 0 \\ -1 & 0 & c & 0 & 1 & 0 \\ -b & 0 & bc & c & 0 & 1 \end{array}\right] \begin{array}{c} \\ \times(-1) \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & b & 1 & 0 & 0 \\ 1 & 0 & -c & 0 & -1 & 0 \\ -b & 0 & bc & c & 0 & 1 \end{array}\right] \begin{array}{c} \\ \\ +b\mathrm{II} \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & b & 1 & 0 & 0 \\ 1 & 0 & -c & 0 & -1 & 0 \\ 0 & 0 & 0 & c & -b & 1 \end{array}\right].$$

The left half of the rref is not I_3, hence the matrix is not invertible, whatever b and c are.