Mathematics 201, Spring 207, Assignment #3

1. a. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_m$ be a 3 dimensional vector then we can represent a matrix A as $A = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m]$ Since A is a 3×3 matrix. Then we see

$$F_{1,2} A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \end{bmatrix} = \begin{bmatrix} v_{1,2} & v_{2,2} & \cdots & v_{m,2} \\ v_{1,1} & v_{2,1} & \cdots & v_{m,1} \\ v_{1,3} & v_{2,3} & \cdots & v_{m,3} \end{bmatrix}$$

So we see that the first row has now replaced the second row and vice versa.

b. $f_{1,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $f_{2,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

c. Since $F_{1,2}$ swaps the first two rows, in order to get the original matrix, we must swap the rows again. Therefore, $F_{12} F_{12} = I$ so $F_{12}^{-1} = F_{12}$

2. a. $A\vec{x} = \vec{v}$ and $AR\vec{v} = I_2\vec{v} = \vec{v}$

so we can say $A\vec{x} = \vec{v} = AR\vec{v}$

$$A\vec{x} = AR\vec{v}$$

$$\vec{x} = R\vec{v} \Rightarrow \vec{x} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

b. $R\vec{x} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ so $\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

so we get the system of equations $\begin{cases} 2x_1 + x_2 = 0 \\ x_1 = 1 \\ -x_1 + x_2 = -3 \end{cases}$

if $x_1 = 1$ then $x_2 = -2(1) = -2$

so $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

3. a. $\ker(T) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

So the solutions are vectors of the form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} s+t \\ -s+t \\ t \end{bmatrix} \Rightarrow \begin{cases} x_1 = s+t \\ x_2 = -s+t \\ x_3 = t \end{cases} \Big\} x_1 + x_2 = 2t$$

So $x_1 + x_2 = 2x_3 \Rightarrow x_1 + x_2 - 2x_3 = 0$ are the equations which contain the solution of the kernel.

Therefore, $A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

b. Since the image contains ~~all~~ the non-reduntant columns of the transformation we get: $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

4. To prove W is a linear subspace we must check:

a) W contains the zero vector

b) if $\vec{w}_1 \in W$ and $\vec{w}_2 \in W$ then $\vec{w}_1 + \vec{w}_2 \in W$

c) if $\vec{w}_1 \in W$ and k is a scalar then $k\vec{w}_1 \in W$.

a. $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_1^2 + x_2^2 + 1 = 0 \right\}$

Check: a) if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in W$ then $0^2 + 0^2 + 1 = 0$, but this is not true so this subset does not contain the zero vector.

Not a subspace

b. $\omega = \left\{ t\begin{bmatrix}1\\0\end{bmatrix} \mid t \geq 0 \right\}$

Check: a) if $t=0$ then $0\begin{bmatrix}1\\0\end{bmatrix} = \vec{0} \in \omega$ ✓

b) if $t_1\begin{bmatrix}1\\0\end{bmatrix} + t_2\begin{bmatrix}1\\0\end{bmatrix} = (t_1+t_2)\begin{bmatrix}1\\0\end{bmatrix}$ then $(t_1+t_2) \geq 0$

So $(t_1+t_2)\begin{bmatrix}1\\0\end{bmatrix} \in \omega$

c) $kt\begin{bmatrix}1\\0\end{bmatrix} \notin \omega$ since $kt \geq 0$ only if $k \geq 0$

→ Not closed under scalar multiplication

Not a subspace

c. $\omega = \left\{ \begin{bmatrix}x_1\\x_2\end{bmatrix} \mid x_1x_1 = 0 \right\}$

Check: a) since $\vec{0} = \begin{bmatrix}0\\0\end{bmatrix}$ & $0 \cdot 0 = 0$ then ω contains the zero vector

b) if $\begin{bmatrix}x_1\\x_2\end{bmatrix} \in \omega$ and $\begin{bmatrix}x_1'\\x_2'\end{bmatrix} \in \omega$ then

$\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}x_1'\\x_2'\end{bmatrix} = \begin{bmatrix}x_1'+x_1\\x_2'+x_2'\end{bmatrix} \in \omega$ since $(x_1'+x_1)^2 = 0$ ✓

c) $k\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}kx_1\\kx_2\end{bmatrix}$ so $k^2x_1^2 = k^2(0) = 0$ so $k\begin{bmatrix}x_1\\x_2\end{bmatrix} \in \omega$

A subspace

d. $\omega = \left\{ \begin{bmatrix}x_1\\x_2\end{bmatrix} \mid x_1 - x_2 = 1 \right\}$

Check: a) since $\vec{0} = \begin{bmatrix}0\\0\end{bmatrix}$ and $0 - 0 \neq 1$ then ω does not contain the zero vector

Not a subspace

6. a. $\vec{V} = a\vec{V}_1 + b\vec{V}_2 + c\vec{V}_3$

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1a + 0b + 0c\\ 0a + 1b + 0c\\ 2a + 1b + 0c\\ -1a + 0b + 1c\end{bmatrix} \Rightarrow \begin{cases}1 = 1a\\ 0 = 1b\\ 0 = 2a + 1b\\ 0 = -1a + 1c\end{cases} \quad \begin{array}{|l} a = 1\\ b = 0\\ 2a + 1b = 2 + 0 = 0\\ \text{Not true}\end{array}$$

There is a contradiction therefore, this equation cannot be solved so $\vec{v}$ is not a linear combination of $\vec{v}_2, \vec{v}_3, \vec{v}_4$.

b. $$\begin{bmatrix}1\\2\\4\\-2\end{bmatrix} = \begin{bmatrix}1a\\ 1b\\ 2a + 1b\\ -1a + 1c\end{bmatrix} \Rightarrow \begin{cases}a = 1\\ b = 2\\ 4 = 2(1) + 1(2) = 4 \checkmark\\ -2 = -1(1) + c \Rightarrow c = -1\end{cases}$$

So $\vec{V} = \vec{V}_1 + 2\vec{V}_2 + (-1)\vec{V}_3$ so $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$.

7. a. If the vectors are linearly dependent then we can represent one vector as a linear combination of the other 2. Let's see if we can do this:

$$\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}2\\0\\1\\-1\end{bmatrix} a + \begin{bmatrix}0\\2\\-1\\1\end{bmatrix} b \Rightarrow \begin{cases}1 = 2a & \rightarrow a = 1/2\\ -1 = 2b & \rightarrow b = -1/2\\ 1 = a - b & 1 = (1/2) - (-1/2) = 1 \checkmark\\ -1 = -a + b & -1 = -(1/2) + (-1/2) = -1 \checkmark\end{cases}$$

So $$\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\0\\1\\-1\end{bmatrix} + \left(-\frac{1}{2}\right)\begin{bmatrix}0\\2\\-1\\1\end{bmatrix}$$ so they are linearly dependent

b. $$\begin{bmatrix}2\\0\\-1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} a + \begin{bmatrix}0\\0\\-1\\1\end{bmatrix} b \Rightarrow \begin{cases}2 = a\\ 0 = a\\ -1 = a - b\\ 2 = b\end{cases}$$

Since $a = 2 = 0$ is not possible there is no solution to this therefore, the vectors must be linearly independent

8. a. Since the span of $\mathbb{R}^3$ is e_1, e_2, and e_3, there is no 3x2 matrix which can have an image of 3 vectors.

$\rightarrow$ No A.

b. If you can reduce the matrix to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ or something of this form then you know the kernel will be $\{\vec{0}\}$. However you can also have a non-zero kernel if you have a ~~matrix~~ matrix $\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\rightarrow$ Some A.

c. im(A) is made up of the columns of A since A has two column we know those vectors can either be multiples of each other or not so we cannot say if they're linearly independent.

$\rightarrow$ Some A.

d. Both columns cannot be free columns so we cannot have 2 vectors in the span of the kernel.

$\rightarrow$ No A.

9. The image is made of all non-redundant columns

a. $\text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right)$

b. $\text{im}(A) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right)$

(Since $\begin{bmatrix} -4 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$)

10. a. $\operatorname{rref}\left(\begin{bmatrix}1&2&0\\2&2&-1\end{bmatrix}\right) \xrightarrow{-2\mathrm{I}} \begin{bmatrix}1&2&0\\0&-2&-1\end{bmatrix} \begin{matrix}+\mathrm{II}\\ \times -1/2\end{matrix} \rightarrow \begin{bmatrix}1&0&-1\\0&1&1/2\end{bmatrix}$ ✓

Now that we have the matrix in rref, it will be easier to find the kernel.

$$\begin{bmatrix}1&0&-1\\0&1&1/2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \vec{0} \Rightarrow \begin{Bmatrix}x_1 - x_3 = 0\\ x_2 + \frac{1}{2}x_3 = 0\end{Bmatrix} \Rightarrow \begin{Bmatrix}x_1 = x_3\\ x_2 = -\frac{1}{2}x_3\end{Bmatrix}$$

Now let $x_3 = t$

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}t\\-\frac{1}{2}t\\t\end{bmatrix} = t\begin{bmatrix}1\\-1/2\\1\end{bmatrix}$$

So $\ker(A) = \operatorname{span}\left(\begin{bmatrix}1\\-1/2\\1\end{bmatrix}\right)$

b. The matrix $\begin{bmatrix}1&1&0&3\\0&0&-1&2\end{bmatrix}$ is almost in rref so we will proceed.

$$\begin{bmatrix}1&1&0&3\\0&0&-1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \vec{0} \Rightarrow \begin{Bmatrix}x_1 + x_2 + 3x_4 = 0\\ -x_3 + 2x_4 = 0\end{Bmatrix}$$

So we can write: $\begin{Bmatrix}x_1 = -x_2 - 3x_4\\ x_3 = 2x_4\end{Bmatrix}$

let $x_2 = s$, $x_4 = t$

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-s-3t\\s\\2t\\t\end{bmatrix} = s\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-3\\0\\2\\1\end{bmatrix}$$

So $\ker(A) = \operatorname{span}\left(\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\2\\1\end{bmatrix}\right)$