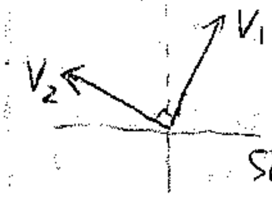
# Solution #5

#1. 

$v_1=\begin{pmatrix}1\\2\end{pmatrix}$ $v_2=\begin{pmatrix}-2\\1\end{pmatrix}$ . $v_1 \perp v_2$

Since $P\begin{pmatrix}1\\2\end{pmatrix}=1\begin{pmatrix}1\\2\end{pmatrix}$ . $P\begin{pmatrix}-2\\1\end{pmatrix}=0\begin{pmatrix}-2\\1\end{pmatrix}$

$\Rightarrow P\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}=\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

$\Rightarrow \mathcal{B}=\{v_1, v_2\}$ is a basis s.t. $[P]_{\mathcal{B}}=\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

#2. (a) Direct calculations.

(b) $J^n = J^{n-4}\cdot J^4 = J^{n-4}\cdot I_2 = J^{n-4}$

$J^{11} = J^{11-4} = J^7 = J^{7-4} = J^3 = -J$

#3. (a) Direct calculations show $AS = SJ = \begin{pmatrix}3 & 2\\1 & 1\end{pmatrix}$

(b) $AS = SJ \Rightarrow A^7 S = SJ^7 = -SJ \Rightarrow A^7 = -SJS^{-1} = -A$

#4. Assume $J \sim \begin{pmatrix}x & 0\\0 & y\end{pmatrix}=K \Rightarrow$ there is an invertible $S=\begin{pmatrix}a & b\\c & d\end{pmatrix}$, $JS=SK$

$$\Rightarrow \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix}=\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x & 0\\0 & y\end{pmatrix} \Rightarrow \begin{cases}ax=c\\cx=-a\\by=d\\dy=-b\end{cases} \Rightarrow \begin{cases}a^2=-c^2\\b^2=-d^2\end{cases}$$

$\Rightarrow a=b=c=d=0 \Rightarrow S$ is not invertible (Contradiction!)

Thus $J$ is not similar to any diagonal matrix.

#5. (a) Since $A \neq 0$, $A^2=0$, $\text{rank}(A)=1 \Rightarrow \dim(\ker A)=1$, $\dim(\text{Im}A)=1$

Since $\forall y \in \text{Im}A$, $y = Ax$ for some $x$, $Ay = A^2x = 0$

$\Rightarrow y \in \text{Ker}(A) \Rightarrow \text{Ker} A \supseteq \text{Im} A$, but they have the same dimension $=1 \Rightarrow \ker A = \text{Im} A$.

(b) Since $\ker A \neq \{0\}$, we assume $v_1 \in \ker A$, $v_1 \neq 0$, $Av_1 = 0$

Using $\ker A = \operatorname{Im} A$, $v_1 \in \operatorname{Im} A \Rightarrow$ There is $v_2 \neq 0$ s.t. $Av_2 = v_1$

To show $\{v_1, v_2\}$ is a basis, it suffices to show $v_1, v_2$ are linearly independent. If $v_2 = cv_1$, $v_2 \in \ker A \Rightarrow Av_2 = 0$

$\Rightarrow cv_1 = 0 \Rightarrow c = 0 \Rightarrow v_2 = 0$ (Contradiction!)

Thus $\{v_1, v_2\}$ is a basis.

(c) $Av_1 = 0$, $Av_2 = v_1 \Rightarrow A[v_1\ v_2] = [v_1\ v_2]\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = [0\ \ v_1]$

$\Rightarrow A$ is similar to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

#6 Let $A = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$, $V = \{S \in \mathbb{R}^{2\times 2} : SA = S\}$

To show $V$ is a subspace of $\mathbb{R}^{2\times 2}$, it suffices to show:

(1) $0 \in V$

(2) $V$ is closed under linear combinations

Indeed, $0A = 0 \Rightarrow 0 \in V$, (1) is true

$(c_1S_1 + c_2S_2)A = c_1S_1A + c_2S_2A = c_1S_1 + c_2S_2 \Rightarrow$ (2) is true

Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $SA = S \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow \begin{cases} a = -2b \\ c = -2d \end{cases} \Rightarrow S = \begin{pmatrix} -2b & b \\ -2d & d \end{pmatrix} = b\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}$

$\Rightarrow \dim V = 2$ and $V = \operatorname{span}\left\{\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}\right\}$

#7 It suffices to show "$I_n$ and $A$ are linearly dependent" $\Leftrightarrow$ "$A = kI_n$ for some $k$"

LHS $\Leftrightarrow c_1I_n + c_2A = 0$ for some $c_1, c_2$ [redacted] $c_2 \neq 0$

$\Leftrightarrow A = -\frac{c_1}{c_2}I_n$ (as $c_2 \neq 0$).

$\Leftrightarrow$ RHS

Let $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$.

#8 (a) $tr(A_1 + A_2) = a_1 + d_1 + a_2 + d_2 = tr(A_1) + tr(A_2)$
$tr(kA_1) = k(a_1 + d_1) = k\,tr(A_1)$. $\Rightarrow$ tr is linear

(b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $tr(A) = 0 \Rightarrow a + d = 0 \Rightarrow d = -a$

$$\Rightarrow A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ is a basis of $\ker(tr)$

#9. T is linear: (1) $T(p_1 + p_2)(x) = p_1(x+2) + p_2(x+2) = Tp_1(x) + Tp_2(x)$
(2) $T(kp)(x) = kp(x+2) = k\,Tp(x)$

T is invertible: $S(p)(x) = p(x-2)$, $ST = TS = id$.
$\Rightarrow$ S is the inverse of T.

#10. (a) $E_A$ is linear: (1) $E_A(p_1 + p_2) = p_1(A) + p_2(A) = E_A(p_1) + E_A(p_2)$
(2) $E_A(kp) = kp(A) = kE_A(p)$

(b) $\forall a_0 \in \mathbb{R}$, $a_0 I_2 = E_A(a_0) \in Im(E_A) \Rightarrow$ ~~dim(EA) ≥ 1~~ $\dim(Im\,E_A) \geq 1$
$\Rightarrow rank(E_A) \geq 1$.

(c) Problem #7 gives: $A \neq kI_2$ for all $k \in \mathbb{R} \Leftrightarrow A$ and $I_2$ are linearly independent
Thus, $\dim span\{I_2, A\} = 2$
$E_A(a_0 + a_1 x) = a_0 I_2 + a_1 A \in Im\,E_A \Rightarrow span\{I_2, A\} \subseteq Im\,E_A$
$\Rightarrow \dim(Im\,E_A) \geq 2 \Rightarrow rank(E_A) \geq 2$

(d) It suffices to show $\dim\ker(E_A) \geq 1$, by Rank-nullity theorem.
Let $A = \begin{pmatrix} b & \\ & c \end{pmatrix}$. $E_A(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 + a_1 b + a_2 b^2 & \\ & a_0 + a_1 c + a_2 c^2 \end{pmatrix} = 0$

$\Rightarrow \begin{cases} a_0 + a_1 b + a_2 b^2 = 0 \\ a_0 + a_1 c + a_2 c^2 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = 0$, $rank\begin{pmatrix} 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \leq 2$

$\Rightarrow \dim\ker\begin{pmatrix} 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \geq 1 \Rightarrow \dim(\ker E_A) \geq 1 \Rightarrow rank(E_A) \leq 2$