Solutions of Homework 6

1. (a) We want to show that there is a unique polynomial $p(x)$ in P_3 such that $p(x_i) = y_i$, $i = 1, 2, 3, 4$, for any prescribed (x_i, y_i), $i = 1, 2, 3, 4$. Consider the linear transformation $T: P_3 \to \mathbb{R}^4$ defined by $T(p(x)) = \begin{bmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \end{bmatrix}$. In relation to T, our goal becomes showing that for any $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ in $\mathbb{R}^4$, there is a unique polynomial $p(x)$ in P_3 such that $T(p(x)) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$. But this is precisely the definition of T being invertible, which is the same as T being an isomorphism. Therefore, our task reduces to showing that T is an isomorphism.

Notice that $\dim P_3 = \dim \mathbb{R}^4 = 4$, so we can show that T is an isomorphism by showing that $\ker(T) = \{0\}$. To find $\ker(T)$, set

$$T(p(x)) = \begin{bmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Recall that a nonzero polynomial of degree at most 3 has no more than 3 roots. But $\begin{bmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ says that $p(x)$ has four distinct roots $x_1 < x_2 < x_3 < x_4$, hence $p(x)$ must be the zero polynomial. Thus $\ker(T) = \{0\}$ and T is an isomorphism.

(b) We want $p_1(x)$ in P_3 such that $\begin{bmatrix} p_1(x_1) \\ p_1(x_2) \\ p_1(x_3) \\ p_1(x_4) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, that is,

$p_1(x_1) = 1$, $p_1(x_2) = 0$, $p_1(x_3) = 0$, $p_1(x_4) = 0$. $p_1(x_2) = p_1(x_3) = p_1(x_4) = 0$ says that x_2, x_3, x_4 are 3 roots of $p_1(x)$. This leads us to considering $(x-x_2)(x-x_3)(x-x_4)$. In order that $p_1(x_1)=1$, we need only divide $(x-x_2)(x-x_3)(x-x_4)$ by the scalar $\frac{1}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$.

Therefore, $p_1(x) = \frac{1}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}(x-x_2)(x-x_3)(x-x_4)$. Note that $p_1(x)$ has degree 3 and is indeed in P_3.

The polynomials $p_2(x)$ satisfying $\begin{bmatrix} p_2(x_1) \\ p_2(x_2) \\ p_2(x_3) \\ p_2(x_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$,

$p_3(x)$ satisfying $\begin{bmatrix} p_3(x_1) \\ p_3(x_2) \\ p_3(x_3) \\ p_3(x_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and

$p_4(x)$ satisfying $\begin{bmatrix} p_4(x_1) \\ p_4(x_2) \\ p_4(x_3) \\ p_4(x_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ can be found in

similar way:

$$p_2(x) = \frac{1}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}(x-x_1)(x-x_3)(x-x_4),$$

$$p_3(x) = \frac{1}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}(x-x_1)(x-x_2)(x-x_4),$$

$$p_4(x) = \frac{1}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}(x-x_1)(x-x_2)(x-x_3).$$

$p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ are linearly independent because their images under T are linearly independent (think about it), hence they form a basis of $\mathbb{R}^4$.

(c) Here $x_1=-2$, $x_2=-1$, $x_3=1$, $x_4=2$. We write out the polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ constructed in (b) explicitly:

$$p_1(x) = \frac{1}{(-1)\cdot(-3)\cdot(-4)}(x+1)(x-1)(x-2) = -\frac{1}{12}x^3+\frac{1}{6}x^2+\frac{1}{12}x-\frac{1}{6},$$

$$p_2(x) = \frac{1}{1\cdot(-2)\cdot(-3)}(x+2)(x-1)(x-2) = \frac{1}{6}x^3-\frac{1}{6}x^2-\frac{2}{3}x+\frac{2}{3},$$

$$p_3(x) = \frac{1}{3\cdot 2\cdot(-1)}(x+2)(x+1)(x-2) = -\frac{1}{6}x^3-\frac{1}{6}x^2+\frac{2}{3}x+\frac{2}{3},$$

$p_4(x) = \frac{1}{4\cdot 3\cdot 1}(x+2)(x+1)(x-1) = \frac{1}{12}x^3 + \frac{1}{6}x^2 - \frac{1}{12}x - \frac{1}{6}$.

We want a polynomial $p(x)$ such that $T(p(x)) = \begin{bmatrix} p(-2) \\ p(-1) \\ p(1) \\ p(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Note that $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \vec{e}_1 + \vec{e}_4$. By linearity of T, if $p(x) = p_1(x) + p_4(x)$, then $T(p(x)) = T(p_1(x)) + T(p_4(x)) = \vec{e}_1 + \vec{e}_4$. Therefore, $p(x) = p_1(x) + p_4(x) = \frac{1}{3}x^2 - \frac{1}{3}$ is the unique polynomial we desire.

- With the basis $\mathcal{B}$ constructed in (b), we can easily write out the polynomial $p(x)$ in P_3 satisfying $\begin{bmatrix} p(-2) \\ p(-1) \\ p(1) \\ p(2) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ (*) for any $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ in $\mathbb{R}^4$. By linearity of T, $p(x) = a p_1(x) + b p_2(x) + c p_3(x) + d p_4(x)$. This is particularly useful if we need polynomials in P_3 satisfying (*) for many different $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$'s.

2. Consider the standard basis $\mathcal{B} = (1, x, x^2)$ of P_2. We work with $\mathcal{B}$-coordinates for this problem. The $\mathcal{B}$-coordinates of $f(x)$, $xf(x)$ and $g(x)$ are $[f(x)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $[xf(x)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $[g(x)]_{\mathcal{B}} = \begin{bmatrix} -k \\ 1-k \\ 1 \end{bmatrix}$. $f(x)$, $xf(x)$, $g(x)$ form a basis of P_2 ($\Leftrightarrow$ $f(x)$, $xf(x)$, $g(x)$ are linearly independent) if and only if their $\mathcal{B}$-coordinates are linearly independent. Therefore, we set up the system $c_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -k \\ 1-k \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

and solve it by Gaussian elimination:

$$\begin{bmatrix} -1 & 0 & -k \\ 1 & -1 & 1-k \\ 0 & 1 & 1 \end{bmatrix} \div(-1) \to \begin{bmatrix} 1 & 0 & k \\ 1 & -1 & 1-k \\ 0 & 1 & 1 \end{bmatrix} -\mathrm{I} \to \begin{bmatrix} 1 & 0 & k \\ 0 & -1 & 1-2k \\ 0 & 1 & 1 \end{bmatrix} \div(-1) \to \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 2k-1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\cdots\Big] -\mathrm{II} \to \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 2k-1 \\ 0 & 0 & 2-2k \end{bmatrix}.$$

The system has only the zero solution $c_1 = c_2 = c_3 = 0$ if and only if

$2-2k \neq 0$, i.e. $k \neq 1$. Thus, $f(x)$, $xf(x)$, $g(x)$ form a basis of P_2 only when $k \neq 1$.

3. Consider the natural basis $\mathfrak{B} = (\cos x, \sin x)$ of V. We work with $\mathfrak{B}$-coordinates for this problem. To find the $\mathfrak{B}$-matrix B of T, recall that the columns of B are the $\mathfrak{B}$-coordinates of the transforms of the elements in $\mathfrak{B}$. Hence we compute

$T(\cos x) = (b-1)\cos x - a\sin x \Rightarrow$ the first column of B is $\begin{bmatrix} b-1 \\ -a \end{bmatrix}$;

$T(\sin x) = a\cos x + (b-1)\sin x \Rightarrow$ the second column of B is $\begin{bmatrix} a \\ b-1 \end{bmatrix}$.

Thus, $B = \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}$.

Now one might find the following fact useful, which is in 2.4 of our textbook. Define the determinant of a 2×2 matrix $\begin{bmatrix} m & n \\ p & q \end{bmatrix}$ by

$\det \begin{bmatrix} m & n \\ p & q \end{bmatrix} = mq - np$. Geometrically, the absolute value of the determinant equals the area of the parallelogram spanned by $\vec{v} = \begin{bmatrix} m \\ p \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} n \\ q \end{bmatrix}$, the two columns of $\begin{bmatrix} m & n \\ p & q \end{bmatrix}$. The useful fact is, $\begin{bmatrix} m & n \\ p & q \end{bmatrix}$ is invertible if and only if its determinant $mq - np \neq 0$.

With the above fact at hand, T is an isomorphism if and only if its $\mathfrak{B}$-matrix B is invertible, which is true if and only if $\det B = (b-1)^2 + a^2 \neq 0$. Therefore, T is an isomorphism only when $a \neq 0$ OR $b \neq 1$. (Note, it should be "or" instead of "and". Think about it.)

$(b-1)^2 + a^2 = 0 \Longleftrightarrow b = 1$ AND $a = 0$.

4. (a) In order that $[T]_{\mathfrak{U}}$ is diagonal,

$T(x + c_1) = x + 1 = k(x + c_1) \Rightarrow c_1 = 1, \quad k = 1$;

$T(x^2 + c_2 x + c_3) = (x+1)(2x + c_2) = 2x^2 + (c_2 + 2)x + c_2 = l(x^2 + c_2 x + c_3)$

$\Rightarrow l = 2, \quad c_2 + 2 = 2c_2, \quad c_2 = 2c_3$

$\Rightarrow c_2 = 2, \quad c_3 = 1$.

The above are obtained by matching the coefficients of like terms.

So $c_1 = 1$, $c_2 = 2$, $c_3 = 1$.

(b) $S_{\mathfrak{B}\to\mathfrak{U}}$ can be constructed column by column; the columns of $S_{\mathfrak{B}\to\mathfrak{U}}$

are the $\mathcal{U}$-coordinate vectors of the elements in $\mathcal{B}$.

$1 = 1 \Rightarrow$ the first column of $S_{\mathcal{B}\to\mathcal{U}}$ is $\begin{bmatrix}1\\0\\0\end{bmatrix}$;

$x = (x+1) - 1 \Rightarrow$ the second column of $S_{\mathcal{B}\to\mathcal{U}}$ is $\begin{bmatrix}-1\\1\\0\end{bmatrix}$;

$x^2 = (x^2+2x+1) - 2(x+1) + 1 \Rightarrow$ the third column of $S_{\mathcal{B}\to\mathcal{U}}$ is $\begin{bmatrix}1\\-2\\1\end{bmatrix}$.

So $$S_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix}1 & -1 & 1\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}.$$

(c) We construct $[T]_{\mathcal{B}}$ column by column.

$T(1) = 0 \Rightarrow$ the first column of $[T]_{\mathcal{B}}$ is $\begin{bmatrix}0\\0\\0\end{bmatrix}$;

$T(x) = x+1 \Rightarrow$ the second column of $[T]_{\mathcal{B}}$ is $\begin{bmatrix}1\\1\\0\end{bmatrix}$;

$T(x^2) = 2x^2+2x \Rightarrow$ the third column of $[T]_{\mathcal{B}}$ is $\begin{bmatrix}0\\2\\2\end{bmatrix}$.

Therefore, $[T]_{\mathcal{B}} = \begin{bmatrix}0 & 1 & 0\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix}$.

By the computation in (a), it is easy to see that

$$[T]_{\mathcal{U}} = \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}, \quad \text{noting } T(1) = 0.$$

The relation among $[T]_{\mathcal{B}}$, $[T]_{\mathcal{U}}$ and $S_{\mathcal{B}\to\mathcal{U}}$ is $[T]_{\mathcal{U}} S_{\mathcal{B}\to\mathcal{U}} = S_{\mathcal{B}\to\mathcal{U}} [T]_{\mathcal{B}}$.
we verify it:

$$\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 1\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 1 & -2\\0 & 0 & 2\end{bmatrix},$$

correct!

5. (a) We construct $[T]_{\mathcal{U}}$ column by column.

$$T\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} - \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

(by inspection or solve a system of equations) $\Rightarrow$ the first column of $[T]_{\mathcal{U}}$ is $\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}$;

$$T\left(\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right) = \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

$\Rightarrow$ the second column of $[T]_{\mathcal{U}}$ is $\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$;

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ the third column of $[T]_{\mathcal{U}}$ is $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$;

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$\Rightarrow$ the fourth column of $[T]_{\mathcal{U}} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ -1 \end{bmatrix}$.

Therefore, $[T]_{\mathcal{U}} = \begin{bmatrix} 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$.

(b) To find a basis of $\operatorname{im}(T)$ and a basis of $\ker(T)$, we first find a basis of $\operatorname{im}([T]_{\mathcal{U}})$ and a basis of $\ker([T]_{\mathcal{U}})$.

Notethat the first column $\vec{v}_1$ and the second column $\vec{v}_2$ of $[T]_{\mathcal{U}}$ are non-redundant columns, because each has a 1 in a component where the other has a 0. Note also that the third column $\vec{v}_3$ and fourth column $\vec{v}_4$ are redundant, with $\vec{v}_3 = -\frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$ (or $\frac{1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2 + \vec{v}_3 = \vec{0}$) and $\vec{v}_4 = -\frac{1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2$ (or $\frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 + \vec{v}_4 = \vec{0}$). The non-redundant columns $\vec{v}_1$, $\vec{v}_2$ form a basis of $\operatorname{im}([T]_{\mathcal{U}})$:

$$\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

and the relations defining the redundant columns give a basis of $\ker([T]_{\mathcal{U}})$: $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$. You might also find these bases by computing $\operatorname{rref}([T]_{\mathcal{U}})$ instead of by inspection.

Transforming the above vectors back to matrices in $\mathbb{R}^{2\times 2}$, we find that a basis of $\operatorname{im}(T)$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix};$$

and a basis of $\ker(T)$ is

$$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$\frac{1}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}+\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\1&0\end{bmatrix}.$$

6. (a) We construct $[T]_{\mathcal{B}}$ column by column.

$$T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)=\begin{bmatrix}1&-1\\1&-1\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\1&0\end{bmatrix}\Rightarrow$$

the first column of $[T]_{\mathcal{B}}$ is $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$;

$$T\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=\begin{bmatrix}1&-1\\1&-1\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&1\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}\Rightarrow$$

the second column of $[T]_{\mathcal{B}}$ is $\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$;

$$T\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=\begin{bmatrix}1&-1\\1&-1\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}-1&0\\-1&0\end{bmatrix}=-\begin{bmatrix}1&0\\0&0\end{bmatrix}-\begin{bmatrix}0&0\\1&0\end{bmatrix}\Rightarrow$$

the third column of $[T]_{\mathcal{B}}$ is $\begin{bmatrix}-1\\0\\-1\\0\end{bmatrix}$;

$$T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}1&-1\\1&-1\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&-1\\0&-1\end{bmatrix}=-\begin{bmatrix}0&1\\0&0\end{bmatrix}-\begin{bmatrix}0&0\\0&1\end{bmatrix}\Rightarrow$$

the fourth column of $[T]_{\mathcal{B}}$ is $\begin{bmatrix}0\\-1\\0\\-1\end{bmatrix}$.

Therefore, $[T]_{\mathcal{B}}=\begin{bmatrix}1&0&-1&0\\0&1&0&-1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}$.

(b) Construct $S_{\mathcal{B}\to\mathcal{U}}$ column by column. The columns are the $\mathcal{U}$-coordinate vectors of elements in $\mathcal{B}$.

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\Rightarrow$$ the first column of $S_{\mathcal{B}\to\mathcal{U}}$ is $\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\0\\0\end{bmatrix}$;

$$\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\Rightarrow$$ the second column of $S_{\mathcal{B}\to\mathcal{U}}$ is $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$;

$$\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\1&0\end{bmatrix}\Rightarrow$$ the third column of $S_{\mathcal{B}\to\mathcal{U}}$ is $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$;

$$\begin{bmatrix}0&0\\0&1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\Rightarrow$$ the fourth column of $S_{\mathcal{B}\to\mathcal{U}}$ is $\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\\0\end{bmatrix}$.

Therefore, $S_{\mathcal{B}\to\mathcal{U}}=\begin{bmatrix}\frac{1}{2}&0&0&\frac{1}{2}\\\frac{1}{2}&0&0&-\frac{1}{2}\\0&1&0&0\\0&0&1&0\end{bmatrix}$.

The relation among $[T]_{\mathcal{B}}$, $[T]_{\mathcal{U}}$ and $S_{\mathcal{B}\to\mathcal{U}}$ is $[T]_{\mathcal{U}} S_{\mathcal{B}\to\mathcal{U}} = S_{\mathcal{B}\to\mathcal{U}} [T]_{\mathcal{B}}$.

we verify it:

$$\begin{bmatrix} 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \text{ right!}$$

7. (a) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ the first column of $S_{\mathcal{U}\to\mathcal{B}}$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$;

$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = -\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ the second column of $S_{\mathcal{U}\to\mathcal{B}}$ is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Therefore, $S_{\mathcal{U}\to\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

(b) $\begin{bmatrix} 3 & 0 & -1 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ the first column

of $[T]_{\mathcal{B}}$ is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$;

$\begin{bmatrix} 3 & 0 & -1 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ the second column of

$[T]_{\mathcal{B}}$ is $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Thus, $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}$.

$\begin{bmatrix} 3 & 0 & -1 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = 4\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \Rightarrow$ the first column of

$[T]_{\mathcal{U}}$ is $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$;

$\begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \Rightarrow$ the second column of

$[T]_{\mathcal{U}}$ is $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

So $[T]_{\mathcal{U}} = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$.

(c) Construct $S_{\mathcal{B}\to\mathcal{U}}$ column by column as in (a) or use the fact that $S_{\mathcal{B}\to\mathcal{U}} = S_{\mathcal{U}\to\mathcal{B}}^{-1}$ to get $S_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. The relation among $[T]_{\mathcal{B}}$, $[T]_{\mathcal{U}}$ and $S_{\mathcal{B}\to\mathcal{U}}$ is $[T]_{\mathcal{U}} S_{\mathcal{B}\to\mathcal{U}} = S_{\mathcal{B}\to\mathcal{U}} [T]_{\mathcal{B}}$, we verify it:

$$\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix}, \text{ correct!}$$

8. For Problems 8 and 10, one might find the following fact useful.
Suppose V is a subspace of $\mathbb{R}^n$, and $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m)$ is a basis of V.
Then $\vec{x}$ in $\mathbb{R}^n$ is orthogonal to V if and only if $\vec{x}$ is orthogonal to all of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m$ in V.
In order that $\vec{x}$ is orthogonal to V, it suffices to require that $\vec{x}$ be orthogonal to all vectors in a basis of V.
This fact is easy to prove. Every vector in V is a linear combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m$. So if $\vec{x}\cdot\vec{v}_i = 0$ for each $i = 1, 2, \cdots, m$, then $\vec{x}\cdot\vec{v} = 0$ for any $\vec{v}$ in V, since dot product is distributive.

By this fact, $\operatorname{span}\{\vec{v}\}^{\perp}$ consists of those vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$ such that $\vec{x}\cdot\vec{v} = 0$, i.e. $x_1 - 2x_2 - x_3 = 0$. Solving this equation, we see that $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $\operatorname{span}\{\vec{v}\}^{\perp}$.

9. (a) Note that $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are already orthogonal to each other. So in order that $\vec{u}_1, \vec{u}_2$ form an orthonormal basis, we only need scaling factors c_1, c_2 so that the lengths of $\vec{u}_1$ and $\vec{u}_2$ are both 1.

$$\left\| \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\| = \sqrt{2^2 + 2^2 + 1^2} = 3 \Rightarrow c_1 = \frac{1}{3};$$

$$\left\| \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\| = \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2} \Rightarrow c_2 = \frac{\sqrt{2}}{2}.$$

(b) For $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$, $\operatorname{proj}_V(\vec{x}) = (\vec{u}_1\cdot\vec{x})\,\vec{u}_1 + (\vec{u}_2\cdot\vec{x})\,\vec{u}_2$

$$= \left(\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} + \left(\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

$$= \left(\frac{2}{3} x_1 + \frac{2}{3} x_2 + \frac{1}{3} x_3 \right) \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} + \left(-\frac{\sqrt{2}}{2} x_1 + \frac{\sqrt{2}}{2} x_2 \right) \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{9} x_1 + \frac{4}{9} x_2 + \frac{2}{9} x_3 \\ \frac{4}{9} x_1 + \frac{4}{9} x_2 + \frac{2}{9} x_3 \\ \frac{2}{9} x_1 + \frac{2}{9} x_2 + \frac{1}{9} x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} x_1 - \frac{1}{2} x_2 \\ -\frac{1}{2} x_1 + \frac{1}{2} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{17}{18} x_1 - \frac{1}{18} x_2 + \frac{2}{9} x_3 \\ -\frac{1}{18} x_1 + \frac{17}{18} x_2 + \frac{2}{9} x_3 \\ \frac{2}{9} x_1 + \frac{2}{9} x_2 + \frac{1}{9} x_3 \end{bmatrix}.$$

Therefore, the matrix of Proj_V is $\begin{bmatrix} \frac{17}{18} & -\frac{1}{18} & \frac{2}{9} \\ -\frac{1}{18} & \frac{17}{18} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{bmatrix}$.

10. By the fact stated in Problem 8, to find $\ker(T)^\perp$, we first need a basis of $\ker(T)$. So set $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 + 3x_4 &= 0 \\ x_1 + x_4 &= 0 \end{aligned} : \quad \left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] - \mathrm{I} \to \left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right] \div 2$$

$$\left[\begin{array}{ccccc} 1 & -2 & 1 & 3 & 0 \\ 0 & 1 & -\frac{1}{2} & -1 & 0 \end{array}\right] + 2\mathrm{II} \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & -1 & 0 \end{array}\right] \to \begin{aligned} x_1 &= -x_4 \\ x_2 &= \tfrac{1}{2} x_3 + x_4. \end{aligned}$$

A basis of $\ker(T)$ is thus $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

$\ker(T)^\perp$ then consists of those vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ in $\mathbb{R}^4$ that are orthogonal to both vectors in the above basis, i.e. those that satisfy

$$\begin{aligned} \tfrac{1}{2} x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_4 &= 0. \end{aligned}$$

Solve this system: $\left[\begin{array}{cccc|c} 0 & \frac{1}{2} & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \end{array}\right] \times 2$

$$\to \left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \end{array}\right] - \mathrm{I} \to \left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 0 \\ -1 & 0 & -2 & 1 & 0 \end{array}\right] \times(-1) \to \left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & -1 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 = -2x_3 + x_4 \\ x_2 = -2x_3 \end{array}$$

A basis of $\ker(T)$ is therefore $\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Book Problems

4.2 22 T is a linear transformation by definition. T fails to be an isomorphism because $\dim P_2 = 3 \neq \dim \mathbb{R} = 1$.

4.2 54 Set $T(at^2+bt+c) = 0$ and get $\frac{35}{3}a + \frac{5}{2}b + 5c = 0 \Rightarrow c = -\frac{7}{3}a - \frac{1}{2}b$.
Thus a typical element in $\ker(T)$ can be expressed as
$at^2 + bt - \frac{7}{3}a - \frac{1}{2}b = a(t^2 - \frac{7}{3}) + b(t - \frac{1}{2})$ where a, b are arbitrary.
So $\ker(T) = \text{span}\{t^2 - \frac{7}{3}, t - \frac{1}{2}\}$, $\text{null}(T) = 2$.
Using constant polynomials for $f(t)$ immediately yields
$\text{im}(T) = \mathbb{R}$, $\text{rank}(T) = 1$.

4.2 64 Note that $\dim P_3 = \dim \mathbb{R}^{2\times 2} = 4$, so P_3 and $\mathbb{R}^{2\times 2}$ are isomorphic and there exists an isomorphism from P_3 to $\mathbb{R}^{2\times 2}$. Consider the standard basis $\mathcal{B} = (1, t, t^2, t^3)$ of P_3 and the standard basis $\mathcal{B}' = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$ of $\mathbb{R}^{2\times 2}$. The most natural isomorphism from P_3 to $\mathbb{R}^{2\times 2}$ send $\mathcal{B}$-coordinate vectors of P_3 to the same $\mathcal{B}'$-coordinate vectors of $\mathbb{R}^{2\times 2}$:

$$T: \underset{\text{in } P_3}{a + bt + ct^2 + dt^3} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ in } \mathbb{R}^{2\times 2}.$$

Look at the following commutative diagram:

$$\begin{tikzcd} \underset{\text{in } P_3}{a + bt + ct^2 + dt^3} \arrow[r, "T"] \arrow[d, "L_{\mathcal{B}}"'] & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ in } \mathbb{R}^{2\times 2} \arrow[dl, "L_{\mathcal{B}'}"] \\ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ in } \mathbb{R}^4 & \end{tikzcd}$$

where $L_{\mathcal{B}}$ and $L_{\mathcal{B}'}$ denote coordinate transformations.

4.3 2 Use coordinates with respect to the standard basis.

$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, $\begin{bmatrix}2\\3\\5\\7\end{bmatrix}$, $\begin{bmatrix}1\\4\\6\\8\end{bmatrix}$ are not linearly independent ($\vec{v}_1 - 4\vec{v}_2 + \vec{v}_3 + \vec{v}_4 = \vec{0}$), so the matrices $\begin{bmatrix}1&1\\1&1\end{bmatrix}$, $\begin{bmatrix}1&2\\3&4\end{bmatrix}$, $\begin{bmatrix}2&3\\5&7\end{bmatrix}$, $\begin{bmatrix}1&4\\6&8\end{bmatrix}$ are not linearly independent.

4.3 8 Note that $U^{2\times2}$ is the space of upper triangular matrices, so its basis consists of 3 matrices. Construct the matrix A of T with respect to the basis $\mathfrak{A}$ column by column as in Problems 4, 5, 6, 7.

$$A = \begin{bmatrix}0&0&0\\2&0&-2\\0&0&0\end{bmatrix}.$$

A is not invertible $\Rightarrow$ T is not an isomorphism.

A basis of $\mathrm{im}(A)$ is $\begin{bmatrix}0\\2\\0\end{bmatrix}$ $\Rightarrow$ a basis of $\mathrm{im}(T)$ is $\begin{bmatrix}0&2\\0&0\end{bmatrix}$;

$\mathrm{rank}(T) = 1$.

A basis of $\ker(A)$ is $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$ $\Rightarrow$ a basis of $\ker(T)$ is

$\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $\begin{bmatrix}0&1\\0&0\end{bmatrix}$; $\mathrm{null}(T) = 2$.

4.3 42 Construct $S = S_{\mathfrak{B}\to\mathfrak{A}}$ column by column as in Problems 4, 5, 6, 7.

$$S = \begin{bmatrix}1&0&1\\0&1&0\\1&0&-1\end{bmatrix}.$$

The matrix B of T with respect to the basis $\mathfrak{B}$ in 4.3 8 is

$B = \begin{bmatrix}0&0&0\\0&0&4\\0&0&0\end{bmatrix}$. We verify $AS = SB = \begin{bmatrix}0&0&0\\0&0&4\\0&0&0\end{bmatrix}$.

To find $S_{\mathfrak{A}\to\mathfrak{B}}$, either construct it column by column or use the fact $S_{\mathfrak{A}\to\mathfrak{B}} = S^{-1}_{\mathfrak{B}\to\mathfrak{A}}$. $S_{\mathfrak{A}\to\mathfrak{B}} = \begin{bmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&1&0\\\frac{1}{2}&0&-\frac{1}{2}\end{bmatrix}$.

4.3 60 $S_{\mathfrak{B}\to\mathfrak{A}} = \begin{bmatrix}1&1\\0&1\end{bmatrix}$, $S_{\mathfrak{A}\to\mathfrak{B}} = \begin{bmatrix}1&-1\\0&1\end{bmatrix}$.

If $\mathfrak{A} = (a_1, a_2, \cdots, a_n)$, $\mathfrak{B} = (b_1, b_2, \cdots, b_n)$ are two bases of a linear space V, then $\mathfrak{A}$ and $\mathfrak{B}$ are related through $S_{\mathfrak{B}\to\mathfrak{A}}$ by $[\, b_1 \ b_2 \ \cdots \ b_n \,] = [\, a_1 \ a_2 \ \cdots \ a_n \,]\, S_{\mathfrak{B}\to\mathfrak{A}}$. This is nothing but a a reformulation of the fact that the columns of $S_{\mathcal{B}\to\mathcal{A}}$ are the $\mathcal{A}$-coordinate vectors of the elements in $\mathcal{B}$. Just do matrix multiplication formally on the right side of this relation to see this.

So in our problem, $[\vec{b}_1 \ \vec{b}_2] = [\vec{a}_1 \ \vec{a}_2] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

5.1 4 $\quad \theta = \arccos \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \arccos \dfrac{9}{85}\sqrt{85}$.

5.1 10 $\quad \vec{u}\cdot\vec{v} = 0 \Rightarrow 2+3k+4=0 \Rightarrow k=-2$.

5.1 16 $\quad$ Assume that $\vec{u}_4 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is such that $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ form an orthonormal basis of $\mathbb{R}^4$. Then

$$\vec{u}_4 \cdot \vec{u}_1 = 0 \Rightarrow \tfrac{1}{2}a + \tfrac{1}{2}b + \tfrac{1}{2}c + \tfrac{1}{2}d = 0,$$
$$\vec{u}_4 \cdot \vec{u}_2 = 0 \Rightarrow \tfrac{1}{2}a + \tfrac{1}{2}b - \tfrac{1}{2}c - \tfrac{1}{2}d = 0,$$
$$\vec{u}_4 \cdot \vec{u}_3 = 0 \Rightarrow \tfrac{1}{2}a - \tfrac{1}{2}b + \tfrac{1}{2}c - \tfrac{1}{2}d = 0.$$

Solving this system, we have $a=d$, $b=-d$, $c=-d$, so

$$\vec{u}_4 = \begin{bmatrix} d \\ -d \\ -d \\ d \end{bmatrix}.$$

Furthermore, $\|\vec{u}_4\| = 1 \Rightarrow 4d^2 = 1 \Rightarrow d = \pm\frac{1}{2}$.

Thus, there are two $\vec{u}_4$ such that $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ form an orthonormal basis of $\mathbb{R}^4$: $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$, which agrees with geometric intuition.