1) We'll do this in general first and then apply our result.

Recall that a set of vectors $\{v_i\}$ is said to be orthonormal if $v_i \cdot v_j = \begin{cases} 1, & i=j \\ 0, & \text{otherwise} \end{cases}$. Using this, and the definition $\|v\| = \sqrt{v \cdot v}$ we have the following lemma.

<u>Lemma:</u> If the vectors $\{v_i\}_1^n$ are orthonormal then for any real numbers $\{\alpha_i\}_1^n$, $\|\sum_{i=1}^n \alpha_i v_i\| = \|\alpha_1 v_1 + \dots + \alpha_n v_n\| = \sqrt{\sum_{i=1}^n (\alpha_i)^2}$

Proof: Expand $\|\sum_{i=1}^n \alpha_i v_i\|^2 = (\sum_{i=1}^n \alpha_i v_i) \cdot (\sum_{j=1}^n \alpha_j v_j)$

$$= \sum_{i=1}^n \alpha_i v_i \cdot (\sum_{j=1}^n \alpha_j v_j)$$

$$= \sum_{\substack{i=1 \\ j=1}}^n \alpha_i \alpha_j \, v_i \cdot v_j \qquad \text{by linearity of } \cdot$$

$$= \sum_{i=1}^n (\alpha_i)^2 \qquad \text{by orthonormality.}$$

In our case then, $\alpha_1 = 4$, $\alpha_2 = -3$, $\alpha_3 = 1$, $\alpha_4 = -3$, $\alpha_5 = -1$

so that $L^2 = 4^2 + 3^2 + 1^2 + 3^2 + 1^2 = 36$ so that $L = 6$.

2a) This entire question is just calculation. See your class notes or textbook (p221) for details.

$u_1 = 1/7 \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$, $v_2^\perp = v_2 - (u_1 \cdot v_2) u_1 = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix} - 0$

$u_2 = 1/7 \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$

b) Similarly: $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3a) Again, more routine computations, see pages 222-223 of your textbook. In this particular case, R is 1×1 with only entry $\|\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\| = 3$ and $Q = 1/3 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

b) $$\begin{bmatrix} 4 & 25 & 0 \\ 0 & 0 & -2 \\ 3 & -25 & 0 \end{bmatrix} = \begin{bmatrix} 4/5 & 0 & 3/5 \\ 3/5 & 0 & -4/5 \\ 0 & -1 & 0 \end{bmatrix}^t \begin{bmatrix} 5 & 5 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$M \quad = \quad Q \quad R$$

4) We will do this in two parts: find ~~a~~ basis, apply G-S to it.

First: If $v \in V$ then $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ has $\underset{\text{(question)}}{v_2 = v_1 + v_3}$ so that

$v = v_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The set $\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{b_2} \right\}$ is thus a basis for V.

Second: $u_1 = b_1/\|b_1\| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $b_2^\perp = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$

so that $u_2 = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ \sqrt{2/3} \end{bmatrix}$ and $\{u_1, u_2\}$ is an answer

5) We see that rref $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that columns 1 and 2 of A will serve as a basis for the image. $\left\{ \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$.

Now we apply G-S to orthonormalise,

$u_1 = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ $\qquad u_2 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ so that $\{u_1, u_2\}$ is an answer.

6) If a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal then we know that $\left\| \begin{bmatrix} a \\ c \end{bmatrix} \right\| = \left\| \begin{bmatrix} b \\ d \end{bmatrix} \right\| = 1$, that is, $a^2 + c^2 = 1 = b^2 + d^2$. A vector of unit length may be written as $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$.

But we also know that $\begin{bmatrix} a \\ c \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = 0$, that is

$0 = \cos\theta \cos\varphi + \sin\theta \sin\varphi = \cos(\theta - \varphi)$. Thus $\theta - \varphi = \pi/2 + n\pi$, for some $n \in \mathbb{Z}$.

Recalling some ancient knowledge, $\cos\varphi = \cos((\theta - \pi/2) - n\pi)$, where if n is even $\cos(\theta - \pi/2 - k2\pi) = \cos(\theta - \pi/2) = \sin\theta$ and $\sin\varphi = \sin(\theta - \pi/2) = -\cos\theta$.

So in one case $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. If n is odd then

$n\pi = \pi + k2\pi$ and $\cos\varphi = \cos(\theta - 3\pi/2 - k2\pi) = -\sin\theta$ and
$\sin\varphi = \sin(\theta - 3\pi/2 - k2\pi) = \cos\theta$ so that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

7a) Recall that $A = (a_{ij})$ is symmetric iff $a_{ij} = a_{ji}$ and skew-symmetric iff $a_{ij} = -a_{ji}$. For symmetric A, B, nA is clearly symmetric as is $nA + B$ (in fact this is a vector subspace of $M_{n\times n}$).

b) Recall that $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ so that

$$(AB - BA)_{ij} = \sum_{k=1}^{n} (a_{ik} b_{kj} - b_{ik} a_{kj})$$
$$= \sum_{k=1}^{n} (a_{ki} b_{jk} - a_{jk} b_{ki}) \quad \text{using } a_{ik} = a_{ki},\ b_{kj} = b_{jk} \text{ etc.}$$
$$= -\sum_{k=1}^{n} (a_{jk} b_{ki} - a_{ki} b_{jk})$$
$$= -(AB - BA)_{ji}$$ so that $AB - BA$ is skew-symmetric

c) I_n, $2A$ are symmetric and so if A^2 is then so is $I_n + 2A + A^2$. To see that A^2 is, $(A^2)_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj} = \sum_{k=1}^{n} a_{ki} a_{jk} = \sum_{k=1}^{n} a_{jk} a_{ki} = (A^2)_{ji}$
thus $I_n + 2A + A^2$ is symmetric.

d) By the above B^2 is symmetric as B is, but the product of two symmetric matrices needn't be symmetric. In fact we see that

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} a_{ki} b_{jk} = \sum_{k=1}^{n} b_{jk} a_{ik} = (BA)_{ji}$$ so if $AB \neq BA$

then we might win. Such an example may be found by setting
$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $B = \begin{bmatrix} 0 & c \\ c & d \end{bmatrix}$, $AB^2 = \begin{bmatrix} ac^2 & acd \\ bcd & bc^2 + bd^2 \end{bmatrix}$. If $acd \neq bcd$ then we win, otherwise we may set $A = 0$ to get a symmetric result.

8) A is similar to B iff there is an invertible C such that $A = CBC^{-1}$. By a result of class (or the book)

$(C^t)^{-1} = (C^{-1})^t$ — to see this $(CC^{-1})^t = (C^{-1})^t C^t$

but $(CC^{-1})^t = I^t = I$

so that $(C^{-1})^t C^t = I$ and similarly $C^t (C^{-1})^t = I$

and thus $(C^{-1})^t = (C^t)^{-1}$.

and so we have

$$A^t = (CBC^{-1})^t = (C^{-1})^t B^t C^t = (C^t)^{-1} B^t C^t$$

So that A^t is similar to B^t as witnessed by C^t.

9a) There is something more general happening – see if you can work it out! In this case, $\|Tv\| = \|v\|$ so we would guess yes, indeed we see that $\frac{1}{5}\begin{pmatrix} 0 & -4 & 3 \\ 0 & 3 & 4 \\ 5 & 0 & 0 \end{pmatrix}$ has the desired properties

But how did we arrive at this?

We desire a mapping $\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \mapsto \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$, or by linearity $\begin{bmatrix} 0 \\ -4/5 \\ 3/5 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. As this is to be orthogonal, its inverse exists and would send $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ -4/5 \\ 3/5 \end{bmatrix}$, that is, $[T]^{-1} = \Big[\underbrace{\begin{smallmatrix} 0 \\ -4/5 \\ 3/5 \end{smallmatrix}}_{u_1} \Big| u_2 \Big| u_3\Big]$

where $\{u_i\}_1^3$ must form an orthonormal basis.

Taking $u_2 = \begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix}$ $u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ gives $[T]^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -4/5 & 3/5 & 0 \\ 3/5 & 4/5 & 0 \end{bmatrix}$

and as T^{-1} is now orthogonal, $T = (T^{-1})^{-1} = (T^{-1})^t$ as above.

b) There is no such orthogonal T, $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 2 \neq 0 = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$.

10a) Observe that $Av = Iv + vv^tv = v + v(v^tv) = v + \|v\|v = (1+\|v\|)v$ and that for any $w \perp v$, $Aw = Iw + v(v^tw) = w$

(really $v \cdot w = v^tw$ as matrices)

Thus if w_1, w_2 form a basis for the perpendicular complement $v^\perp = (\text{span}\{v\})^\perp$ then $\mathcal{B} = \{v, w_1, w_2\}$ are linearly independent and form a basis for $\mathbb{R}^3$ so that in this basis

$$[A]_{\mathcal{B}} = \begin{bmatrix} 1+\|v\| & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $1+\|v\| \neq 0$. Using the change of basis matrix $S^{-1} = [v | w_1 | w_2]$ we see that $A = S[A]_{\mathcal{B}}S^{-1}$.

b) The above gives $AS = S\underbrace{[A]_{\mathcal{B}}}_{D}$ so we must only ensure that S is orthogonal. This is true iff S^{-1} is orthogonal and as $S^{-1} = [v | w_1 | w_2]$ this is true iff $\mathcal{B} = \{v, w_1, w_2\}$ are orthonormal. This needn't be the case but we may always perform G-S to find $\{u_1 = v/\|v\|, u_2, u_3\}$ which are. In this basis, ~~...~~ we see that $A v/\|v\| = v/\|v\| + v(v \cdot v/\|v\|) = v/\|v\| + v\|v\| = v/\|v\|(1+\|v\|^2)$
$(v \cdot v = \|v\|^2)$

$= u_1(1+\|v\|^2)$ so that

$$[A]_{\mathcal{B}} = \begin{bmatrix} 1+\|v\|^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is similar to A via $S = [u_1 | u_2 | u_3]$ an orthogonal matrix.