LINEAR ALGEBRA (MATH 110.201)

MIDTERM I

Name: ____________________

Section number/TA: ____________________

Instructions:

(1) Do not open this packet until instructed to do so.
(2) This midterm should be completed in **50 minutes**.
(3) Notes, the textbook, and digital devices **are not permitted**.
(4) Discussion or collaboration is **not permitted**.
(5) All solutions must be written on the pages of this booklet.
(6) Justify your answers, and write clearly; points will be subtracted otherwise.

Exercise	Points	Your score
1	5	5
2	5	5
3	5	5
4	5	5
5	5	5

Exercise 1 (5 points) Let a, b be fixed real numbers. Consider the following system of equations:

$$\begin{aligned} X + Y &= a \\ X + 2Y + Z &= b \\ X + 3Y + 2Z &= 0 \end{aligned}$$

Determine all possible values of a, b for which the above system has a solution. When the system has a solution, describe all solutions in terms of a and b.

Solution:

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 2 & 1 & b \\ 1 & 3 & 2 & 0 \end{array}\right] & \begin{array}{c} \\ -\mathrm{I} \\ -\mathrm{I} \end{array} \end{array} \rightarrow \begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b-a \\ 0 & 2 & 2 & -a \end{array}\right] & \begin{array}{c} -\mathrm{II} \\ \\ -2\mathrm{II} \end{array} \end{array} \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2a-b \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & a-2b \end{array}\right]$$

System has a solution if and only if $a = 2b$. In that case, the solutions are

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} s+3b \\ -s-b \\ s \end{bmatrix}$$ where s can be any number.

Exercise 2 (5 points) Suppose that $T, U : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations. Let c be a fixed real scalar. Consider the function $H : \mathbb{R}^n \to \mathbb{R}^m$ defined by $H(x) = cT(x) + U(x)$. Explain why H is a linear transformation. How does the matrix of H relate to the matrices of T and U?

Solution:

Note that T and U are linear transformations. So

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) \quad \text{for all } \vec{v}, \vec{w} \text{ in } \mathbb{R}^n$$

$$T(k\vec{v}) = kT(\vec{v}) \quad \text{for all } \vec{v} \text{ in } \mathbb{R}^n \text{ and } k \text{ scalar}$$

$$U(\vec{v} + \vec{w}) = U(\vec{v}) + U(\vec{w}) \quad \text{for all } \vec{v}, \vec{w} \text{ in } \mathbb{R}^n$$

$$U(k\vec{v}) = kU(\vec{v}) \quad \text{for all } \vec{v} \text{ in } \mathbb{R}^n \text{ and } k \text{ scalar}$$

Therefore, for any $\vec{v}$, $\vec{w}$ in $\mathbb{R}^n$ and scalar k,

$$\begin{aligned} H(\vec{v} + \vec{w}) &= cT(\vec{v} + \vec{w}) + U(\vec{v} + \vec{w}) \\ &= c(T(\vec{v}) + T(\vec{w})) + (U(\vec{v}) + U(\vec{w})) \\ &= (cT(\vec{v}) + U(\vec{v})) + (cT(\vec{w}) + U(\vec{w})) \\ &= H(\vec{v}) + H(\vec{w}) \end{aligned}$$

$$\begin{aligned} H(k\vec{v}) &= cT(k\vec{v}) + U(k\vec{v}) \\ &= ckT(\vec{v}) + kU(\vec{v}) \\ &= k(cT(\vec{v}) + U(\vec{v})) \\ &= kH(\vec{v}). \end{aligned}$$

Hence, H is a linear transformation.

$[H] = c[T] + [U]$.

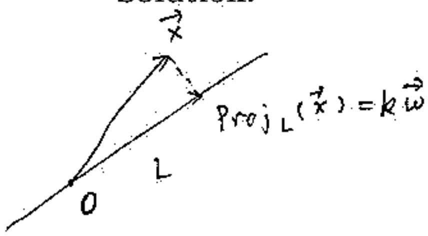
Exercise 3 (5 points) Let $w = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, and let $\mathcal{L}$ be the line in $\mathbb{R}^3$ passing through the origin and w. Let $\mathrm{Proj}_{\mathcal{L}} : \mathbb{R}^3 \to \mathbb{R}^3$ be the function that sends a vector x to its orthogonal projection onto $\mathcal{L}$.

(1) By definition, $\mathrm{Proj}_{\mathcal{L}}(x) = kw$ for some scalar k; express this scalar in terms of x and w.

(2) Prove, using your answer from (1), that $\mathrm{Proj}_{\mathcal{L}}$ is a linear transformation.

(3) Write down the matrix of $\mathrm{Proj}_{\mathcal{L}}$.

Solution:



(1) From the picture, $(\vec{x} - k\vec{w}) \perp \vec{w} \Rightarrow$

$(\vec{x} - k\vec{w}) \cdot \vec{w} = 0 \Rightarrow \vec{x} \cdot \vec{w} = k\, \vec{w} \cdot \vec{w}$.

Therefore, $k = \dfrac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}$.

(2) $\mathrm{Proj}_L(\vec{x}) = k\vec{w} = \dfrac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$. Then it is easy to see from the properties of dot product that

$$\mathrm{Proj}_L(\vec{x} + \vec{y}) = \frac{(\vec{x} + \vec{y}) \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

$$= \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \cdot \vec{w} + \frac{\vec{y} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

$$= \mathrm{Proj}_L(\vec{x}) + \mathrm{Proj}_L(\vec{y})$$

$$\mathrm{Proj}_L(k\vec{x}) = \frac{(k\vec{x}) \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = k \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

$$= k\, \mathrm{Proj}_L(\vec{x})$$

for all $\vec{x}, \vec{y}$ in $\mathbb{R}^3$ and scalars k.

Thus Proj_L is a linear transformation

(3) Suppose $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $\mathrm{Proj}_L(\vec{x}) = \dfrac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$

$$= \frac{x_1 + 2x_2 + 2x_3}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{9}x_1 + \frac{2}{9}x_2 + \frac{2}{9}x_3 \\ \frac{2}{9}x_1 + \frac{4}{9}x_2 + \frac{4}{9}x_3 \\ \frac{2}{9}x_1 + \frac{4}{9}x_2 + \frac{4}{9}x_3 \end{bmatrix}$$

So matrix

$$\begin{bmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix}.$$

Exercise 4 (5 points) Let a and b denote real numbers, with $a \neq 0$. Determine whether the following matrix is invertible, and write down A^{-1} if it is.

$$A = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$$

Double-check that your answer is correct when $a = 1$ and $b = 1$.

Solution:

~~[illegible]~~ Note that $a \neq 0$, so we can divide by a.

$$\begin{bmatrix} a & 0 & b & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 1 \end{bmatrix} \div a \longrightarrow \begin{bmatrix} 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 1 \end{bmatrix} \div a$$

$$\rightarrow \begin{bmatrix} 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 1 \end{bmatrix} \div a \rightarrow \begin{bmatrix} 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix}\Bigg] -\frac{b}{a}\mathrm{III} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{a} & 0 & -\frac{b}{a^2} & 0 \\ 0 & 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 1 \end{bmatrix} \div a \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{a} & 0 & -\frac{b}{a^2} & 0 \\ 0 & 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{a} \end{bmatrix}$$

$$\begin{matrix} 0 \\ 0 \\ 0 \\ \frac{1}{a} \end{matrix}\Bigg] -\frac{b}{a}\mathrm{IV} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{a} & 0 & -\frac{b}{a^2} & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{a} & 0 & -\frac{b}{a^2} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{a} \end{bmatrix}$$

The left half of the last matrix is I_4, so A is invertible, and

$$A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & -\frac{b}{a^2} & 0 \\ 0 & \frac{1}{a} & 0 & -\frac{b}{a^2} \\ 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & \frac{1}{a} \end{bmatrix}.$$

Exercise 5 (5 points) Let A be an $n \times n$ matrix. Recall that

$$\ker(A) = \{x \in \mathbb{R}^n \mid Ax = 0_n\}$$

where 0_n is the vector in $\mathbb{R}^n$ having all 0's as coordinates. Show that if $x \in \ker(A)$, then $x \in \ker(A^2)$ (where $A^2 = A \cdot A$ is the matrix product of A with itself).

Solution:

$$\vec{x} \in \ker(A) \Rightarrow A\vec{x} = \vec{0} \Rightarrow A^2\vec{x} = A(A\vec{x})$$

$$= A\vec{0} = \vec{0} \Rightarrow \vec{x} \in \ker(A^2).$$