LINEAR ALGEBRA (MATH 110.201)

MIDTERM II

Name: ______________________________

Section number/TA: ______________________________

Instructions:

(1) Do not open this packet until instructed to do so.
(2) This midterm should be completed in **50 minutes**.
(3) Notes, the textbook, and digital devices **are not permitted**.
(4) Discussion or collaboration is **not permitted**.
(5) All solutions must be written on the pages of this booklet.
(6) Justify your answers, and write clearly; points will be subtracted otherwise.

Exercise	Points	Your score
1	5	
2	5	
3	5	
4	5	
5	5	

Exercise 1 (5 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

(1) Find a basis for the subspace $\mathrm{Ker}(A) \subseteq \mathbb{R}^4$. What is $\dim(\mathrm{Ker}(A))$?
(2) Find a basis for the subspace $\mathrm{Im}(A) \subseteq \mathbb{R}^2$. What is $\dim(\mathrm{Im}(A))$?

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{array}{l} \\ -5\mathrm{I} \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{bmatrix} \begin{array}{l} \\ \div(-4) \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{array}{l} -2\mathrm{I} \\ \\ \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

(1) The solution of the system $A\vec{x} = \vec{0}$ is $\begin{array}{l} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 . \end{array}$

or $\begin{bmatrix} s + 2t \\ -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$, where s and t are arbitrary

Therefore, a basis of $\ker(A)$ is $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$, and

$\dim \ker(A) = 2$.

(2) Pick columns in A corresponding to those in $\mathrm{rref}(A)$ that contain leading 1's:

a basis of $\mathrm{im}(A)$ is $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$;

$\dim \mathrm{im}(A) = 2$.

Exercise 2 (5 points) Let V be a vector space. Suppose that $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are all nonzero scalars. Show that if $v_1, \ldots, v_n \in V$ are linearly independent, then $\lambda_1 v_1, \ldots, \lambda_n v_n$ are also linearly independent. Will this conclusion still be true if we allow some of the $\lambda_1, \ldots, \lambda_n$ to be zero? Explain why or why not.

Solution:

Form the equation

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_n \lambda_n v_n = 0.$$

Since $v_1, v_2, \cdots, v_n$ are linearly independent, we must have $c_1 \lambda_1 = 0$, $c_2 \lambda_2 = 0$, $\cdots$, $c_n \lambda_n = 0$.

Because $\lambda_1, \lambda_2, \cdots, \lambda_n$ are all nonzero scalars, it follows that $c_1 = 0$, $c_2 = 0$, $\cdots$, $c_n = 0$. Therefore, $\lambda_1 v_1, \lambda_2 v_2, \cdots \lambda_n v_n$ are linearly independent.

If we allow some of the $\lambda_1, \lambda_2, \cdots, \lambda_n$ to be zero, the conclusion will not necessarily be true. This is simply because, if say $\lambda_i = 0$, then $\lambda_i v_i = 0$ and $\lambda_i v_i$ is redundant in the list $\lambda_1 v_1, \lambda_2 v_2, \cdots, \lambda_n v_n$.

Note: some of the "0" in the above solution mean the neutral element in V, while the other "0" indicate the scalar 0. Make sure you understand which is what.

Exercise 3 (5 points) Consider the following vectors as points in $\mathbb{R}^2$:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

The goal of this exercise is to find a quadratic polynomial g whose graph passes through these points (i.e., $g(1) = 1$, $g(2) = 2$, and $g(3) = -3$). Consider the following polynomials:

$$f_1(X) = \frac{(X-2)(X-3)}{(1-2)(1-3)} \quad f_2(X) = \frac{(X-1)(X-3)}{(2-1)(2-3)} \quad f_3(X) = \frac{(X-1)(X-2)}{(3-1)(3-2)}$$

(1) Show that f_1, f_2, f_3 are linearly independent in $P_2(\mathbb{R})$. Hint: Note that if $c_1 = 1$, $c_2 = 2$, and $c_3 = 3$, then $f_i(c_i) = 1$ and $f_i(c_j) = 0$ if $i \neq j$.

(2) Deduce that f_1, f_2, f_3 are a basis of $P_2(\mathbb{R})$.

(3) Use (2) to find a polynomial $g \in P_2(\mathbb{R})$ such that $g(1) = 1$, $g(2) = 2$, and $g(3) = -3$.

Solution:

(1) Form the equation $a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) = 0$.
Substituting $x=1$, $x=2$ and $x=3$ in this equation and noting the hint, we obtain

$a_1 = 0$ (consequence of plugging in $x=1$)
$a_2 = 0$ (consequence of plugging in $x=2$)
$a_3 = 0$ (consequence of plugging in $x=3$).

Therefore, f_1, f_2, f_3 are linearly independent in $P_2(\mathbb{R})$.

(2) Since $\dim P_2(\mathbb{R}) = 3$ and f_1, f_2, f_3 are linearly independent, f_1, f_2, f_3 form a basis of $P_2(\mathbb{R})$.

(3) $g(x)$ can be expressed as $g(x) = d_1 f_1(x) + d_2 f_2(x) + d_3 f_3(x)$ for some scalars d_1, d_2, d_3.
Substituting $x=1$, $x=2$, $x=3$ in the expression and noting again the hint, we have

$g(1) = d_1 = 1$,
$g(2) = d_2 = 2$,
$g(3) = d_3 = -3$.

Therefore, $g(x) = f_1(x) + 2f_2(x) - 3f_3(x)$
$= -3x^2 + 10x - 6$.

Exercise 4 (5 points) Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix}$$

(1) What is the dimension of $W = \mathrm{Span}(v_1, v_2)$? (Explain carefully).
(2) Find an orthonormal basis of W.

Solution:

(1) obviously, $\vec{v}_1$ is not the zero vector and $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$. so $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. Hence, $\vec{v}_1$, $\vec{v}_2$ form a basis of span $(\vec{v}_1, \vec{v}_2)$ $\Rightarrow$ dim span $(\vec{v}_1, \vec{v}_2) = 2$.

(2) Perform the Gram-Schmidt process to the basis $\vec{v}_1$, $\vec{v}_2$ of span $(\vec{v}_1, \vec{v}_2)$.

$$\|\vec{v}_1\| = \sqrt{11} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{11}}\vec{v}_1 = \frac{1}{\sqrt{11}}\begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix},$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix} - \left(\frac{1}{\sqrt{11}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix}\right)\frac{1}{\sqrt{11}}\begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix} + \frac{1}{11}\begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 14/11 \\ 12/11 \\ 1 \\ 54/11 \end{bmatrix},$$

$$\|\vec{v}_2^{\perp}\| = \sqrt{\frac{307}{11}},$$

$$\vec{u}_2 = \sqrt{\frac{1}{3377}}\begin{bmatrix} 14 \\ 12 \\ 1 \\ 54 \end{bmatrix}.$$

Therefore, an orthonormal basis of span $(\vec{v}_1, \vec{v}_2)$ is

$$\frac{1}{\sqrt{11}}\begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \sqrt{\frac{1}{3377}}\begin{bmatrix} 14 \\ 12 \\ 1 \\ 54 \end{bmatrix}.$$

Exercise 5 (5 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal linear transformation.

(1) Show that if $x \in \mathbb{R}^n$ and $x \neq \vec{0}_n$, then $T(x) \neq \vec{0}_n$.
(2) Show that if $x, y \in \mathbb{R}^n$, then the dot products $x \cdot y = T(x) \cdot T(y)$ are equal. Hint: You can use the fact that $T(e_1), \ldots, T(e_n)$ is an orthonormal basis of $\mathbb{R}^n$.
(3) Show, using (2), that if $x, y \in \mathbb{R}^n$ are nonzero vectors with angle θ between them, then the angle between $T(x)$ and $T(y)$ is equal to θ.

Solution:

(1) $\vec{x} \neq \vec{0} \Rightarrow \|\vec{x}\| > 0$. Then $\|T(\vec{x})\| = \|\vec{x}\| > 0$ since T is an orthogonal transformation. So $T(\vec{x}) \neq \vec{0}$.

(2) Express $\vec{x} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + \cdots + a_n \vec{e}_n$,
$\vec{y} = b_1 \vec{e}_1 + b_2 \vec{e}_2 + \cdots + b_n \vec{e}_n$,
with appropriate scalars $a_1, \cdots, a_n, b_1, \cdots, b_n$.
Then $\vec{x} \cdot \vec{y} = (a_1 \vec{e}_1 + a_2 \vec{e}_2 + \cdots + a_n \vec{e}_n) \cdot (b_1 \vec{e}_1 + b_2 \vec{e}_2 + \cdots + b_n \vec{e}_n)$
$= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$, by distributivity of the dot product and the fact that $\vec{e}_i \cdot \vec{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

By Linearity of T,
$T(\vec{x}) = a_1 T(\vec{e}_1) + a_2 T(\vec{e}_2) + \cdots + a_n T(\vec{e}_n)$,
$T(\vec{y}) = b_1 T(\vec{e}_1) + b_2 T(\vec{e}_2) + \cdots + b_n T(\vec{e}_n)$.
Since T is an orthogonal transformation, $T(\vec{e}_1), T(\vec{e}_2), \cdots, T(\vec{e}_n)$ is an orthonormal basis of $\mathbb{R}^n$, satisfying
$$T(\vec{e}_i) \cdot T(\vec{e}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$
Then distributivity of dot product yields again
$$T(\vec{x}) \cdot T(\vec{y}) = (a_1 T(\vec{e}_1) + a_2 T(\vec{e}_2) + \cdots + a_n T(\vec{e}_n)) \cdot (b_1 T(\vec{e}_1) + b_2 T(\vec{e}_2) + \cdots + b_n T(\vec{e}_n))$$
$$= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$
Therefore, $\vec{x} \cdot \vec{y} = T(\vec{x}) \cdot T(\vec{y})$.

(3) The angle between $\vec{x}$ and $\vec{y}$ is $\arccos \dfrac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$;
the angle between $T(\vec{x})$ and $T(\vec{y})$ is $\arccos \dfrac{T(\vec{x}) \cdot T(\vec{y})}{\|T(\vec{x})\| \|T(\vec{y})\|}$.
Since $\vec{x} \cdot \vec{y} = T(\vec{x}) \cdot T(\vec{y})$ by (2), $\|T(\vec{x})\| = \|\vec{x}\|$, $\|T(\vec{y})\| = \|\vec{y}\|$, it follows that the two angles are equal.