Math 201
Spring 2014
Midterm 2
04/09/14
Lecturer: Jesus Martinez Garcia
Time Limit: 50 minutes

Name (Print): ______________________

Teaching Assistant ______________________

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a theorem of lemma you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Do not write in the table to the right.

1. (25 points) **Inner Product spaces and Gram-Schmidt.** Recall $C[-1,1]$ denotes all functions which are continuous on $[-1,1]$. Let $P_2 \subset C[-1,1]$, the space of polynomials of degree at most 2. Find an orthonormal basis for P_2 under the inner product given by:

$$\langle f, g\rangle = \frac{1}{2}\int_{-1}^{1} f(t)g(t)dt.$$

(Hint: you may apply Gram-Schmidt using inner product instead of dot product, as we did for Fourier approximations).

2. (25 points) **Linear transformations.** Recall that P is the space of polynomials in one variable. Let $T: P \to P$ be defined by $T(f(t)) = f''(t) + 2f'(t)$.

   (a) (10 points) Show that T is linear.

For all $f(t)$, $g(t)$ in P and all scalars k,

$T(f(t)+g(t)) = (f(t)+g(t))'' + 2(f(t)+g(t))'$

$= (f''(t) + 2f'(t)) + g''(t) + 2g'(t)$

$= T(f(t)) + T(g(t))$;

$T(kf(t)) = (kf(t))'' + 2(kf(t))'$

$= k f''(t) + k\, 2f'(t)$

$= k\, T(f(t))$.

Therefore, T is linear.

   (b) (15 points) Find a basis for Ker(T).

Let $f(t) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial in P. Then $f'(t) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2a_2 x + a_1$,

and $f''(t) = n(n-1) a_n x^{n-2} + (n-1)(n-2) a_{n-1} x^{n-3} + \cdots + 6a_3 x + 2a_2$.

Set $T(f(t)) = f''(t) + 2f'(t) = 0$ and collect terms:

$2n a_n x^{n-1} + [2(n-1) a_{n-1} + n(n-1) a_n] x^{n-2} +$

$[2(n-2) a_{n-2} + (n-1)(n-2) a_{n-1}] x^{n-3} + \cdots + (4a_2 + 6a_3)x +$

$(2a_1 + 2a_2) = 0$.

This yields the system

$$
\begin{aligned}
&2n a_n = 0\\
&2(n-1) a_{n-1} + n(n-1) a_n = 0\\
&2(n-2) a_{n-2} + (n-1)(n-2) a_{n-1} = 0\\
&\vdots\\
&4a_2 + 6a_3 = 0\\
&2a_1 + 2a_2 = 0
\end{aligned}
$$

Solving this system from the first equation all the way to the last, we obtain successively

$a_n = 0$, $a_{n-1} = 0$, $a_{n-2} = 0$, $\cdots$, $a_2 = 0$, $a_1 = 0$.

The only coefficient that is arbitrary is a_0. Therefore, ker(T) consists of constant polynomials $\{a_0 : a_0$ can be any real number$\}$.

3. (25 points) **The matrix of a linear transformation and dimension of vector spaces**

   (a) (10 points) Find the matrix of the linear transformation $L\colon \mathrm{Mat}_2(\mathbb{R}) \to \mathrm{Mat}_2(\mathbb{R})$ given by $L(A) = A^t - 2A$ with respect to the basis

$$\mathfrak{B} = \left\{ v_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ v_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ v_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Construct the matrix, also denoted by L for convenience, of the linear transformation L column by column.

$$L(v_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^t - 2\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & -1 \end{pmatrix} = -2\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$\Longrightarrow$ the first column of L is $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$;

$$L(v_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^t - 2\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$\Rightarrow$ the second column of L is $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$;

$$L(v_3) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^t - 2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$\Longrightarrow$ the third column of L is $\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$;

$$L(v_4) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^t - 2\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow$$

   (b) (15 points) Show that if W is a subspace of V and V is finite dimensional, then W must be finite dimensional as well.

the fourth column of L is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$.

Thus, $L = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

(b) Let m be the maximal number of linearly independent vectors that can be drawn from W (if W were infinite dimensional, then $m = \infty$), and let $v_1, v_2, \cdots, v_m$ be a list of linearly independent vectors in W (if W were infinite dimensional, we have an infinite list $v_1, v_2, v_3, \cdots$ of linearly independent vectors).

Denote $\dim V = n$.

Since W is a subspace of V, $v_1, v_2, \cdots v_m$ is also a list of linearly independent vectors in V. But then $m \le n$, hence W is finite dimensional.

4. (25 points) **Determinants and orthogonal matrices.**

(a) (10 points) Compute

$$\det\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & -1 & 0 & 1\\ 0 & 0 & 1 & -1\\ 0 & -1 & 1 & 2\end{pmatrix}$$

(b) (15 points) For each of the following statements decide if it is true or false. If it is true, prove it. If it is false, provide a counter-example.

(i) If A is a symmetric square matrix of size n and S is an orthogonal matrix of size n, then $S^{-1}AS$ is symmetric.

(ii) If A is an square matrix of size n such that $\|A\vec{u}\| = 1$ for all unit vectors $\vec{u}$, then A is orthogonal.

(iii) The map $\det\colon \mathrm{Mat}_n(\mathbb{R}) \to \mathbb{R}$ is linear.

(i) True. Since S is an orthogonal matrix, $S^{-1} = S^T$, and since A is symmetric, $A^T = A$. Then $S^{-1}AS = S^TAS$.

$(S^{-1}AS)^T = (S^TAS)^T = S^TA^T(S^T)^T = S^TAS = S^{-1}AS$.

Therefore, $S^{-1}AS$ is symmetric.

(ii) True. $A\vec{0} = \vec{0} \Rightarrow \|A\vec{0}\| = \|\vec{0}\|$.

For any nonzero vector $\vec{x}$, $\frac{\vec{x}}{\|\vec{x}\|}$ is a unit vector and

$\vec{x} = \frac{\vec{x}}{\|\vec{x}\|}\|\vec{x}\|$. Thus, $A\vec{x} = A\left(\|\vec{x}\|\frac{\vec{x}}{\|\vec{x}\|}\right) =$

$\|\vec{x}\| A\left(\frac{\vec{x}}{\|\vec{x}\|}\right)$ since A is linear.

Then $\|A\vec{x}\| = \left\|\|\vec{x}\| A\left(\frac{\vec{x}}{\|\vec{x}\|}\right)\right\| = \|\vec{x}\|\left\|A\left(\frac{\vec{x}}{\|\vec{x}\|}\right)\right\|$

$= \|\vec{x}\|$, noting that $\frac{\vec{x}}{\|\vec{x}\|}$ is a unit vector and A preserves length of unit vectors.

Therefore, A preserves length of all vectors $\Rightarrow$ A is orthogonal.