Math 286: Topics in Differential Geometry
Compactness Properties of Minimal Surfaces in Three-Manifolds
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CHAPTER 1

Introduction

1. Introduction

We will interested in the following compactness questions for submanifolds of a fixed manifold M:

(1) Given a sequence Σ_i of (smoothly immersed) submanifolds in fixed complete Riemannian manifold (M, g), to what extent can we say that, up to passing to a subsequence, the Σ_i converge to a limit Σ?
(2) If the Σ_i share property $\mathcal{P}$ to what extent does this affect the answer and to what extent does Σ have property $\mathcal{P}$?

Implicit in any answer is an imposition of a topology on the space of submanifolds together with a restriction on the allowed Σ_i. Additionally, the limit Σ may then only be a "generalized" submanifold and only satisfy $\mathcal{P}$ in "generalized" sense. There are many reasons to be interested in such questions. On the the one had, compactness type arguments often lead to very simple proofs by contradiction and more general often provide a key step in many proofs. On a more fundamental level, when considering a class of submanifolds, useful information can often be found about the class by considering to what extent the class is compact and to analyze how the failure of compactness.

We illustrate with some simple examples. Let us suppose that $\Omega \subset M$ is a smooth open region in M. In addition, assume that the Σ_i are a sequence of submanifolds of fixed dimension, k, with the property that $\bar{\Sigma}_i \backslash \Sigma \cap \Omega = \emptyset$, i.e. each Σ_i is a closed subset in the induced topology on Ω. We consider first a very weak notion of compactness that we will more or less always have. Namely, Hausdorff convergence for closed sets. Indeed, for each Σ_i we introduce the following function on Ω:

$$D_i(p) = d^M(p, \Sigma_i) := \inf \left\{ d^M(p,q) : q \in \Sigma^i \right\}.$$

Here d^M is the distance function on M. One verifies that d_i is always a Lipschitz map with Lipschitz norm at most 1. Moreover, $\Sigma_i = \{p : D_i(p) = 0\}$. It is a standard fact in real analysis that a sequence of bounded Lipschitz maps with bounded Lipschitz constant on a fixed compact set are compact in with respect to Lipschitz convergence. In particular, by a diagonalization argument, up to passing to a subsequence, there is a Lipschitz function D (possibly taking the value ∞) on Ω so that the D_i converges uniformly on compact subsets to D. One then takes $\Sigma = \{p : D(p) = 0\}$. Notice that we needed very few assumptions on the Σ_i, thought, as a price, the convergence is extremely weak and the limit object Σ has no geometry to speak of.

We next consider a slightly more geometric situation. Namely, suppose one has the following uniformity assumption on the Σ_i:

$$Vol^k(\Sigma_i \cap K) \leq C(K) < \infty$$

for all $K \subset\subset \Omega$ a compact subset of Ω and C is a constant depending only on the set K. Given one of the Σ_i, one can readily construct a Radon measure, μ_i, on Ω so that $\operatorname{supp} \mu_i = \Sigma$ and so that $\mu_i(K) = Vol^k(\Sigma_i \cap K)$. The uniformity assumption then implies that the μ_i have a uniform mass bound on compact subsets of Ω. Hence, by standard functional analysis results, up to passing to a subsequence, the μ_i weakly converge to a Radon measure μ on Ω. It is then natural to take $\Sigma = \operatorname{supp} \mu$, though without further assumptions on the sequence and further analysis very little can be said about the geometric structure (if any) of Σ. What is known as Geometric Measure Theory is a development of ideas of this sort and while central to the issues at hand will be mostly avoided in this course.

We conclude with an example where, in contrast to the above two examples, we will get an explicitly geometric object in the limit. To that end, suppose that the Σ_i have uniform curvature bounds on compact subsets of Ω. That is, if $|A_i|$ is the norm of the second-fundamental form of Σ_i then we assume :

$$\sup_{K \cap \Sigma_i} |A_i| \leq C(K) < \infty$$

again for all $K \subset\subset \Omega$ a compact subset of Ω and C is a constant depending only on the set K. In this case, as we will see, up to passing to a subsequence, the surfaces Σ_i converge in a $C^{1,\alpha}$ sense to a union (not necessarily finite) of $C^{1,\alpha}$ submanifolds. Here we are free to choose $0 < \alpha < 1$. We will discuss later the exact nature of the convergence. If Σ_i also have the uniform mass bounds of the preceding example, then one can actually conclude that inside any compact subset of Ω the limit Σ consists of a *finite* union of of $C^{1,\alpha}$ immersed submanifolds, though without this assumption Σ could potentially fill space.

The main thrust of this class will be to understand to what extent one can weaken the uniform curvature bound of last example, but still get a geometric object in the limit. A natural way to approach this is to replace control on the full second fundamental form with control only on part of it. That is on $H = \operatorname{tr} A$ the mean curvature. Indeed, we will discuss only the case where we have maximal control on the mean curvature, in other words when H is identically zero, i.e. when the Σ_i are *minimal submanifolds*. The reason for doing so are three-fold. First of all, this is the setting in which the deepest theorems can be shown and with the least technical effort. Additionally, the minimal case provides a model for more general situations where the mean curvature is only controlled in some sense. Finally, understanding such compactness properties can in turn be used to understand properties of the minimal surfaces themselves.

We will only consider minimal *surfaces* in three-manifolds, this is both for simplicity and because in this setting powerful techniques can be brought to bear on the problem. As we will see, there are many interesting compactness phenomena for minimal surfaces. For instance, for sequences of minimal surfaces with uniform bounds on the L^2 norm of the curvature:

$$\int_{\Sigma_i} |A|^2 \leq C$$

we will obtain smooth convergence away from a finite set of points and the limit object will be smooth. Due to the scaling invariance such a bound is critical for surfaces. It is also natural in the sense that when the surfaces are closed and embedded and M has positive Ricci curvature then this bound automatically follows from a topological bound – i.e. a genus bound. Strikingly, in such manifolds, a result of Choi and Schoen, shows that such a genus bound automatically gives smooth compactness.

The result of Choi and Schoen, exploits properties of minimal surfaces in three-manifolds with positive Ricci curvature quite heavily. However, as all three-manifolds are locally modelled on $\mathbb{R}^3$, it is interesting to consider the question there. By considering catenoids, any attempt to directly extend their result to $\mathbb{R}^3$ can be seen to be impossible (of course there are no closed minimal surfaces in $\mathbb{R}^3$). Nevertheless, the failure of smooth convergence for sequences of embedded minimal surfaces with a genus bound can be understood. The most far reaching results of this nature are due to Colding and Minicozzi. In a series of papers they study the failure of smooth convergence for sequences of embedded minimal surfaces in $\mathbb{R}^3$ with bounded genus. There work is most easily digested when one considers the case of minimal disks. Roughly speaking, they characterize the failure of smooth compactness for embedded minimal disks in $\mathbb{R}^3$, with the failure being always modelled on the helicoid. There proof exploits many properties of embedded minimal surfaces in $\mathbb{R}^3$. One of the most striking is the so-called one-sided curvature estimate, which they prove, this gives a uniform curvature bound for an embedded minimal disk that is close to, and on one side of, a plane.

2. Notation and Conventions

We record here some notation and conventions that will be used throughout these notes. We first note that we will always consider *surfaces* Σ in complete Riemannian three-manifolds (M, g). For simplicity we will always take Σ to be oriented and denote by $\mathbf{n}$ the unit normal field to Σ. We will often consider $(M, g) = (\mathbb{R}^3, g_{euc})$ Euclidean three space and when we do so we will denote the standard coordinates by x_1, x_2 and x_3. We denote by g^Σ the metric induced on Σ and by A^Σ and H^Σ the second fundamental form and mean curvature of Σ with respect to $\mathbf{n}$. For a fixed point in p we will denote by $B_r(p)$ the open metric ball in M of radius r. For $p \in \Sigma$ we denote by $\mathcal{B}_r^\Sigma(p)$ the open metric ball (with respect to g^Σ) of radius r. We denote by $\Sigma_{p,r}$ the component of $B_r(p) \cap \Sigma$ that contains p.

CHAPTER 2

Background

1. Geometry of Graphs

We record here some simple geometric computations for surfaces that are graphs of functions. Suppose $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function. The surface Σ is said to be the graph of u, denoted by Γ_u, provided

$$\Sigma = \Gamma_u := \left\{ (x, y, u(x)) \in \mathbb{R}^3 : (x, y) \in \Omega \right\}. \tag{1.1}$$

Σ has a natural coordinate system given by x and y and it is straightforward to calculate that in these coordinates

$$g^{\Sigma} = (1 + u_x^2) dx^2 + 2u_x u_y dxdy + (1 + u_y^2) dy^2. \tag{1.2}$$

The upward pointing normal

$$\mathbf{n} = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}} \tag{1.3}$$

and with respect to this normal the second fundamental form is given by

$$A^{\Sigma} = \frac{1}{\sqrt{1 + |\nabla u|^2}} \left(u_{xx} dx^2 + 2u_{xy} dxdy + u_{yy} dy^2 \right). \tag{1.4}$$

Combining (1.2) and (1.4) allows one to compute H^{Σ} and to estimate $|A^{\Sigma}|^2$:

$$H^{\Sigma} = \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \tag{1.5}$$

and

$$\frac{|\nabla^2 u|^2}{(1 + |\nabla u|^2)^3} \leq |A^{\Sigma}|^2 \leq 2 \frac{|\nabla^2 u|^2}{(1 + |\nabla u|^2)^3}. \tag{1.6}$$

2. Geometric Arzela-Ascoli Theorem

We begin with a proof of geometric version of the Arzela-Ascoli theorem. For simplicity we work in $\mathbb{R}^3$. A key part of the proof is a simple technical fact that is of independent interest:

LEMMA 2.1. *Suppose that $\Sigma \subset B_{4s}(0) \subset \mathbb{R}^3$ is an immersed smooth surface with $\partial \Sigma \subset \partial B_{4s}$ and that*

$$16 s^2 \sup_{\Sigma} |A_{\Sigma}|^2 \leq 1 \tag{2.1}$$

If $x \in B_{2s} \cap \Sigma$ then $\Sigma_{x,s}$ may be written as a graph of a function u over $T_x \Sigma$ and so that

$$|\nabla u| + s |\nabla^2 u| \leq 2. \tag{2.2}$$

Proof. By a homothetic rescaling we may take $s = 1$. Now fix a point $x \in B_2 \cap \Sigma$. As Σ is smooth and $\Sigma_{x,1}$ is compact there is a uniform $\delta > 0$ (in principal depending on Σ) so that for each point $y \in \Sigma_{x,1}$ so that $\Sigma_{y,\delta}$ can be written as the graph of a function u^y over $T_y\Sigma$ where $|\nabla u^y| \leq 1$. The claim will be shown provided we guarantee that $T_y\Sigma$ doesn't vary too much for $y \in \Sigma_{y,s}$.

To that end, we note that since $d^\Sigma(p,q) \leq |p-q|$ for $p, q \in \Sigma$ we have that $\mathcal{B}_2^\Sigma(x) \subset B_2(x) \cap \Sigma$. A simple geometric computation gives that:

$$|\nabla_X \mathbf{n}| \leq |A^\Sigma||X|$$

In particular, if $p \in \mathcal{B}_2^\Sigma(x)$ and γ is the minimizing geodesic connecting p to x parameterized by arc-length and $L = d^\Sigma(p,x)$ then

$$|\mathbf{n}(p) - \mathbf{n}(q)| \leq \int_0^L |\nabla_{\gamma'} \mathbf{n}| ds \leq \frac{1}{2}.$$

As a consequence, at least for $y \in \mathcal{B}^\Sigma(x)$, the planes $T_y\Sigma$ don't vary too much and so we can write $\mathcal{B}_2^\Sigma(x)$ as the graph over $T_x\Sigma$ of a function u satisfying $|\nabla u| \leq 1$.

The lemma will be shown if we can ensure that $\Sigma_{x,1} \subset \mathcal{B}_2^\Sigma(x)$. Note that the gradient estimate for u and (1.2) gives the bound for $y \in \partial \mathcal{B}_2^\Sigma(x)$

$$|x-y| \geq \frac{1}{\sqrt{2}} d^\Sigma(x,y) = \sqrt{2} > 1.$$

In particular, $\partial \mathcal{B}_2^\Sigma(x) \cap B_1(x) = \emptyset$. This implies that $\Sigma_{x,1} \subset \mathcal{B}_2^\Sigma(x)$ which together with (1.6) proves the claim. □

We are now in a position to show the geometric Arzela-Ascoli theorem. In order to avoid certain technical annoyances, we assume a local mass bound.

Theorem 2.2. *Fix $1 > \alpha > 0$. Let $\Omega \subset \mathbb{R}^3$ be a smooth open region. Suppose that Σ_i is a sequence of (immersed) smooth surfaces in Ω with $\left(\overline{\Sigma} \backslash \Sigma\right) \cap \Omega = \emptyset$ and that satisfy:*

$$Area(\Sigma_i \cap K) \leq C(K) \tag{2.3}$$

and

$$\sup_{\Sigma_i \cap K} |A^\Sigma| \leq C(K) \tag{2.4}$$

for each $K \subset\subset \Omega$. Then there is an $\mathcal{S} = \left\{\Gamma^j\right\}_{j \in I}$ a collection of $C^{1,\alpha}$ (immersed) surfaces Γ^j where

1. *Each Γ^j satisfies $\left(\overline{\Gamma}^j \backslash \Gamma^j\right) \cap \Omega = \emptyset$;*
2. *The set $\Sigma = \cup_{j \in I} \Gamma^j$ is closed in the relative topology of Ω;*
3. *For each compact subset $K \subset \Omega$ at most a finite number of Γ^j meet K.*

Moreover, up to passing to a subsequence, the Σ_i converge to Σ in a $C^{1,\alpha}$ sense and with finite multiplicity. Precisely, for each point $x \in \Sigma$ there is a radius $r = r(x) > 0$ so:

1. *$B_{8r}(x) \subset \Omega$;*
2. *For any $y \in B_{2r}(x) \cap \Gamma^j$, $B_{4r}(y) \cap \Gamma^j$ is the graph over $T_y\Gamma^j$ of some $C^{1,\alpha}$ function.*

(3) *For each $\epsilon > 0$ there is a an $i_0 = i_0(x, \epsilon)$ so that for $i > i_0$ the following holds: Any component, $\hat{\Sigma}$, of $\Sigma_i \cap B_{2r}(x)$ that meets $B_r(x)$ can be written as the graph of a function $\hat{u}$ over $T_y\Gamma^j$ for some $y \in \Gamma^j \cap \overline{B}_r(x)$. Moreover, if u is the function whose graph over $T_y\Gamma^j$ gives Γ^j then*
$$|\hat{u} - u|_{1,\alpha} < \epsilon;$$
(4) *The number of components of $B_{2r}(x) \cap \Sigma_i$ meeting $B_r(x)$ is uniformly bounded by $N = N(x)$.*

PROOF. We begin by choosing K_i an exhaustion of Ω by compact sets (so $K_i \subset \mathring{K}_{i+1}$). Let us denote by C_i the corresponding constants given by (2.3) and (2.4) for each K_i. Pick $\delta_i > 0$ so for each point $x \in K_i$ one has $B_{\delta_i}(x) \subset K_{i+1}$ and so that $16\delta_i^2 C_{i+1}^2 \leq 1$. Set $r(x) = \frac{1}{8}\delta_1$ if $x \in K_1$ and for $x \in K_i \backslash K_{i-1}$ set $r(x) = \frac{1}{8}\delta_i$.

By the compactness of closed sets in the Hausdorff topology, up to passing to a subsequence, we may assume the Σ_i converge in the Hausdorff topology to a closed set Σ. Our goal is to show that Σ has the additional structure claimed and that the convergence is also as claimed. By standard covering arguments we can choose a countable number of points, $x_l \in \Sigma$, so that $B_{\frac{1}{5}r(x_l)}(x_l)$ are pairwise disjoint while $B_{r(x_l)}(x_l)$ cover Σ'. Moreover, by compactness of K_i, there will only be a finite number of x_l inside of each K_i.

Fix a x_l in our set of points and let $\rho = 2r(x_l)$. By Lemma 2.1 each component of $\Sigma_i \cap B_{2\rho}(x_l)$ is the graph of some function over some plane in $\mathbb{R}^3$ and moreover, the function has gradient bounded by 1. This implies that $\frac{1}{10}\rho^2$ provides a uniform lower bound on the area of any of these components that meet $B_\rho(x_l)$. In particular, by the uniform area bound there are at most $N = N(x_l)$ of these components. Up to passing to a subsequence we may then assume there are exactly N of them. We will denote these as $\Sigma_i^1, \ldots, \Sigma_i^N$

Now choose points $y_i^1, \ldots, y_i^N$ with $y_i^j \in \Sigma_i^j$, the compactness of $\overline{B}_\rho(x_l)$ and of the unit sphere, gives that up to passing to a subsequence, the points $y_i^j \to y^j \in \Sigma \cap \overline{B}_\rho(x_l)$ and $\mathbf{n}(y_i^j) \to \mathbf{n}^k$. If P^j is the plane through y^j normal to $\mathbf{n}^j$ then by 2.1 for i sufficiently large, we can write each component Σ_i^j as the graph of a function u_i^j over P^j. Moreover, one will have
$$|u_i^j|_{2,0} \leq 3$$
and $u_j^i(y^j) \to 0$ as $i \to \infty$. By the Arzela-Ascoli theorem we then can pass to a subsequence and see that each u_i^j converges to a $C^{1,\alpha}$ function u^j in the $C^{1,\alpha}$ topology. The Hausdorff convergence implies that
$$\Sigma \cap B_\rho(x_l) = \left(\bigcup \Gamma_{u^j}\right) \cap B_\rho(x_l).$$
To get the convergence described in the statement of the theorem we need only ignore the y^j that do not lie in $\overline{B}_r(x_l)$.

By diagonalizing, we can ensure this occurs for each x_l. Notice, this implies that locally Σ is the union of $C^{1,\alpha}$ surfaces. The proof is concluded by verifying that the local surfaces can be glued together in a consistent manner for different x_l, which we leave as an exercise. □

We remark that a simple consequence of the convergence is that for any Γ^j in $\mathcal{S}$ and any $U \subset \Gamma^j$ a bounded open set in Γ^j (with respect to d^{Γ^j}) and any $\epsilon > 0$

less then the focal radius of Γ^j, there is an $i_0 = i_0(U, \epsilon)$ so that for $i > i_0$ at least one component $\hat{\Sigma}_i$ of $\Sigma_i \cap T_\epsilon(U)$ may be written as a normal exponential graph of a function $C^{1,\alpha}$ function u_i defined on U. That is:

$$\hat{\Sigma}_i = \left\{ \mathbf{x}(p) + u_i \mathbf{n}^{\Gamma^j}(p) : p \in U \right\} \subset \mathbb{R}^3$$

where here $\mathbf{x}$ is the position function on Γ^j. The uniform curvature bounds ensure that the Γ^j have a uniform focal radius on compact subsets. In particular, if γ is null-homologous closed curve in a Γ^j then it can be lifted to a null-homologous closed curve in Σ_i for i sufficiently large. In other words, the topology of the sequence influences the topology of the limit.

After studying some properties of minimal surfaces, we will return to this theorem and see what can be shown when we assume that, in addition to the curvature estimate, that the Σ_i are embedded minimal surfaces.

3. Variational Properties of Minimal Surfaces

Recall that we say a smooth surface $\Sigma \subset M$ is *minimal* if it has vanishing mean curvature. Here the mean curvature, H^Σ is the trace of the second fundamental form of Σ that is at a point $p \in \Sigma$,

$$H^\Sigma := \sum_{i=1}^{2} A^\Sigma(E_i, E_i)$$

where here E_i is a local orthonormal frame on Σ near p. The second fundamental form A^Σ measures the discrepency between the Levi-Civita connection of g, ∇^M, on M and of the Levi-Civita connection ∇^Σ of the metric g^Σ, i.e.

$$A^\Sigma(X, Y) = -g(\nabla^M_X Y - \nabla^\Sigma_X Y, \mathbf{n}^\Sigma).$$

The second fundamental form also measures how normal deformations of a hypersurface effect the induced metric. Minimal surfaces arise naturally as the stationary points of the area functional. Precisely, fix a smooth open region Ω in M and suppose that $\left(\overline{\Sigma} \backslash \Sigma\right) \cap \Omega = \emptyset$ we say that Σ is *stationary for area in* Ω if the following holds: For each $U \subset \Omega$ open and with $\overline{U}$ compact in Ω and each smooth family of surfaces Σ_t so that $\Sigma_0 = \Sigma$ and $\Sigma_t = \Sigma$ outside of U

$$\left.\frac{d}{dt}\right|_{t=0} Area(\Sigma_t \cap \overline{U}) = 0$$

We point out that if Σ is embedded and each surface in the deformation Σ_t is embedded and disjoint then the deformation is generated by a smooth compactly supported vector field X on M. Conversely, any such vector field may be integrated to give such a deformation of surfaces – namely take such an X let ϕ_t be the one parameter family of diffeomorphisms generated by X and take $\Sigma_t = \phi_t(\Sigma)$.

To verify that minimal surfaces and stationary surfaces are equivalent, we must compute the first variation of area. To that end let us parameterize Σ by a smooth map

$$F_0 : \Sigma \to M$$

a variation of Σ can then be parameterized by a smooth family of smooth maps

$$F : \Sigma \times (-\epsilon, \epsilon) \to M$$

if we set $F_t(p) = F(p,t)$ then we can take $\Sigma_t = F_t(\Sigma)$. We note that outside of a compact set in Σ, $F_t(p) = F(p)$. In particular, the vector field along F

$$X = \frac{\partial}{\partial t} F$$

restricted to $\Sigma \times \{0\}$ has compact support. Strictly speaking, X is a section of the pull-back bundle F^*TM however as mentioned when each Σ_t is embedded and disjoint we may identify X with a vector field on Σ. Let us write $g(t) = g^{\Sigma_t}$ and after picking local coordinates x_i on Σ we also consider the matrix

$$\begin{aligned} g_{i,j}(t) &= g^{\Sigma_t}(\partial_{x_i} F(\cdot, t), \partial_{x_j} F(\cdot, t)) \\ &= g(\partial_{x_i} F(\cdot, t), \partial_{x_j} F(\cdot, t)) \end{aligned}$$

the volume form of of Σ_t is as

$$d\mu_t = \frac{\sqrt{\det g_{i,j}(t)}}{\sqrt{\det g_{i,j}(0)}} d\mu_0.$$

Hence,

$$\begin{aligned} \frac{d}{dt}|_{t=0} Area(\Sigma_t \cap \bar{U}) &= \frac{d}{dt}|_{t=0} \int_{\Sigma \cap \bar{U}} \frac{\sqrt{\det g_{i,j}(t)}}{\sqrt{\det g_{i,j}(0)}} d\mu_0 \\ &= \int_{\Sigma \cap \bar{U}} \frac{d}{dt}|_{t=0} \frac{\sqrt{\det g_{i,j}(t)}}{\sqrt{\det g_{i,j}(0)}} d\mu_0 \\ &= \int_{\Sigma \cap \bar{U}} \frac{1}{2} \mathrm{tr}\, \frac{d}{dt}|_{t=0} g_{i,j}(t) d\mu_0 \end{aligned}$$

where we have used that for any matrix valued function $A(t)$, $\frac{d}{dt} \log \det A(t) = \mathrm{tr}\, A'(t)$. We compute

$$\begin{aligned} \frac{d}{dt}|_{t=0} g_{i,j}(t) &= \partial_t g(\partial_{x_i} F, \partial_{x_j} F)|_{t=0} \\ &= g(\nabla^M_X \partial_{x_i} F, \partial_{x_j} F) + g(\partial_{x_i} F, \nabla^M_X \partial_{x_j} F) \\ &= g(\nabla^M_{\partial_{x_i} F} X, \partial_{x_j} F) + g(\partial_{x_i} F, \nabla^M_{\partial_{x_j} F} X) \\ &= 2g(X, \mathbf{n}) A^{\Sigma}(\partial_{x_i} F, \partial_{x_j} F) + g(\nabla^{\Sigma}_{\partial_{x_i} F} X^{\top}, \partial_{x_j} F) + g(\nabla^{\Sigma}_{\partial_{x_j} F} X^{\top}, \partial_{x_i} F) \end{aligned}$$

Where here we used the fact that ∂_t and ∂_{x_i} are coming from coordinates. Here $X^{\top} = X - g(X, \mathbf{n})\mathbf{n}$ is the projection of X to $T\Sigma$, i.e the tangential part of the variation. We let $X^{\perp} = g(X, \mathbf{n})\mathbf{n}$ denote the normal part. As a consequence, taking the the trace and plugging back into our formula yields the first variation formula:

$$\begin{aligned} \frac{d}{dt}|_{t=0} Area(\Sigma_t \cap \bar{U}) &= \int_{\Sigma} H^{\Sigma} g(X, \mathbf{n}) + \mathrm{div}^{\Sigma} X^{\top} d\mu_0 \\ &= \int_{\Sigma} H^{\Sigma} g(X, \mathbf{n}) d\mu_0 \end{aligned}$$

where we used the fact that $X^{\top}$ had compact support and Stokes' theorem. Recall, the divergence of a vector field tangent to Σ is defined at a point by

$$\mathrm{div}^{\Sigma} X^{\top} = \sum_{i=1}^{2} g^{\Sigma}(\nabla^{\top}_{E_i} X^{\top}, E_i).$$

As any function $\phi \in C^\infty_0(\Sigma)$ gives rise to a variation Σ_t with normal part $\phi\mathbf{n}$, an immediate consequence of the first variation formula is that the mean curvature of a surface vanishes identically when and only when the surface is stationary. Another way to think about the first variation formula is to note that for X a vector field of M along Σ that we can define for a point p in Σ

$$\mathrm{div}^{\,\Sigma} X = \sum_{i=1}^{2} g(\nabla_{E_i} X, E_i)$$

where E_i is a orthonormal frame around p. One can then compute that

$$\mathrm{div}^{\,\Sigma} = \mathrm{div}^{\,\Sigma} X^\top + g(X, \mathbf{n}) H^\Sigma.$$

In other words, the first variation formula has the compact form

$$\frac{d}{dt}|_{t=0} Area(\Sigma_t \cap \bar{U}) = \int_\Sigma \mathrm{div}^{\,\Sigma} X d\mu_0$$

We see also that Σ is minimal if and only if

$$\mathrm{div}^{\,\Sigma} X = \mathrm{div}^{\,\Sigma} X^\top$$

for all vector fields X of M along Σ.

Having characterized minimal surfaces as critical points of the area functional it is natural to try and understand what geometric role the index of the critical point plays. That is we wish to look at the second variation of area around a minimal surface. The computation is straightforward, though somewhat involved so I refer the details to [**1**] (see also [**?**]). Given the set-up above, one computes that:

$$\frac{d^2}{dt^2}|_{t=0} Area(\Sigma_t) = \int_\Sigma |\nabla^\Sigma \phi|^2 - |A^\Sigma|^2 \phi^2 - Ric^M(\mathbf{n}, \mathbf{n}) \phi^2 d\mu_0$$

where here $\phi = g(X, \mathbf{n})$. I want to point out that we get such a nice formula, due to our standing assumption that Σ is an oriented hyper-surface, we get a global section of the normal bundle of Σ which trivializes it (i.e. $\mathbf{n}$). More generally, the normal bundle could be much more complex which leads to a messier formula and greatly complicates the analysis. Indeed, this is one of the first and most essential reasons why the theory of minimal hyper-surfaces is so much better understood then for other co-dimensions.

When the second variation is always positive then Σ is a local for minimum for area, i.e. any deformation of Σ increases (at least for a small bit) increases the area of Σ. Note large variations of the surface may in fact have less area, i.e. a stable surface need not be *area-minimizing*. More precisely, consider $U \subset \Sigma$ a smooth open domain, we say that Σ is *stable* in U if for any variation Σ_t with support in U one has

$$\frac{d^2}{dt^2}|_{t=0} Area(\Sigma_t) \geq 0.$$

When Σ is stable one has the following stability estimate for all $\phi \in C^\infty_0(U)$

$$\int_\Sigma |\nabla^\Sigma \phi|^2 d\mu_0 \geq \int_\Sigma |A^\Sigma|^2 \phi^2 + Ric^M(\mathbf{n}, \mathbf{n}) \phi^2 d\mu_0$$

Notice that in Euclidean space, the stability inequality gives a weak bound on the second fundamental form. As we will see, this gives important regularity property for stable minimal surfaces and understanding this will be one of the main goals of the first few weeks of the course.

We note that one of the reasons the theory is so nice under our standing assumptions is that they allow us to most directly interact with spectral theory of self-adjoint operators. Namely, if we let L be the formally self-adjoint Schrödinger operator

$$L = \Delta^{\Sigma} + |A|^2 + Ric^M(\mathbf{n}, \mathbf{n})$$

which we refer to as the stability operator. The left-hand side of the second variation formula is exactly the quadratic form associated to the self-adjoint operator L i.e.

$$\frac{d^2}{dt^2}|_{t=0} Area(\Sigma_t) = -\int_{\Sigma} \phi L \phi d\mu_0.$$

In other, words there is a relationship between the the stability properties of Σ and the spectral properties of L. When $\overline{U}$ is compact then L – thought of as an operator on $H^1_0(U)$ – has discrete spectrum and if Σ is stable then $\lambda_1(U)$ the first dirichlet eigenvalue of L on U is non-negative. We call the number of negative eigenvalues of $-L$ the Morse index of Σ in U. This measures the number of directions one can move the surface in that decreases its area.

We will return to this connection shortly and prove use it to understand the geometric properties of stable surfaces.

4. Minimal Surfaces in $\mathbb{R}^3$

Let us consider now the case of $M = \mathbb{R}^3$ with the Euclidean metric. We begin with some examples of minimal surfaces that will serve as guides to the theory and illustrate its richness. One thing that is useful to keep in mind is that rigid motions and homotheties preserve the minimal surface equation. We also restrict attention to embedded surfaces.

(1) Planes. These are totally geodesic. As we will see they are the only stable complete surfaces.

(2) The cateonid. This is the surface of revolution of the catenary. It is topologically an annulus and unstable (though of Morse index one). Rescalings of the catenoid provide a model of the failure of smooth convergence for minimal surfaces. It is explicitly parameterized by

$$(t, \theta) \mapsto (\cosh t \cos\theta, \cosh t \sin\theta, t)$$

(3) The helicoid. This is a surface described by fixing a line and then looking at the surface swept out by translating an orthogonal line along this axis while rotating it at a constant speed. It has infinite Morse index. As we will see it is the model for the model for the failure of smooth compactness for embedded minimal disks. It is explicitly parameterized by

$$(t, s) \mapsto (s \cos t, s \sin t, t).$$

One useful way to think of the surface is that away from the x_3-axis it is the union of the graphs of the "multi-valued" functions θ and $\theta + \pi$ where here θ is defined to be the angle function for polar coordinates on the x_1-x_2 plane.

(4) Riemann's examples. This is a one parameter family of surfaces consisting of an infinite number of parallel planes joined by small necks distributed along a line transverse to the planes. The parameter corresponds to the angle of the line. This surface has infinite topology. By a scaling one can vary the parameter while keeping the neck sizes fixed and study the the

limits as the parameter degenerates. In the direction corresponding to the line becoming parallel to the planes the surface converges to a catenoid. In the other direction, corresponding to the line becoming perpendicular to the planes, the limit is singular.

A particularly useful property of a minimal surface Σ in $\mathbb{R}^3$ is that the coordinate functions restrict to be a harmonic function on Σ and this characterizes minimal surfaces in $\mathbb{R}^3$.

PROPOSITION 2.3. *Let Σ be a smooth surface in $\mathbb{R}^3$. Then the functions $x_i = x_i|_\Sigma$ satisfy*

$$\Delta^\Sigma x_i = 0$$

if and only if Σ is minimal.

PROOF. For any vector field X of $\mathbb{R}^3$ along Σ we recall that

$$\operatorname{div}^\Sigma X = \operatorname{div}^\Sigma X^\top + g(X, \mathbf{n}) H^\Sigma$$

For any function $\nu \in C_0^\infty(\Sigma \backslash \partial\Sigma)$ this together with $\operatorname{div}^\Sigma \mathbf{e}_i = 0$ implies that

$$\begin{aligned}\int_\Sigma g^\Sigma(\nabla^\Sigma \nu, \nabla^\Sigma x_i) &= \int_\Sigma g(\nabla^\Sigma \nu, \mathbf{e}_i)\\ &= \int_\Sigma \operatorname{div}^\Sigma(\nu \mathbf{e}_i)\\ &= \int_\Sigma \nu H g(\mathbf{n}, \mathbf{e}_i)\end{aligned}$$

where the last equality used Stoke's theorem and (4). Since ν and $i = 1, 2, 3$ are arbitrary this implies that Σ is minimal if and only if the x_i are weakly harmonic and hence actually harmonic. □

REMARK 2.4. If we let $\mathbf{x}(p) = (x_1(p), x_2(p), x_3(p))$ be the position vector of $p \in \Sigma$ then the preceding computation implies that

$$\Delta^\Sigma \mathbf{x} = -H^\Sigma \mathbf{n}$$

A useful consequence of this is the following convex hull principle:

PROPOSITION 2.5. *Suppose that Σ is a smooth and compact minimal surface then*

$$\Sigma \subset Conv(\partial\Sigma)$$

where $Conv(\partial\Sigma)$ is the convex hull of $\partial\Sigma$.

PROOF. For each fixed vector $\mathbf{V}$ in $\mathbb{R}^3$ the function $\phi_\mathbf{V} = g(\mathbf{V}, \cdot)$ is the linear combination of coordinate functions so is harmonic on Σ. Each half-space of $\mathbb{R}^3$ can be represented by $\{\phi_\mathbf{V} \geq h\}$ and so the classical maximum principal implies that if $\partial\Sigma$ is contained in a half-space so is Σ. Since the convex hull of a set is the intersection of the half-spaces containing the set this proves the claim. □

A useful corollary of this is

COROLLARY 2.6. *Let Σ be a smooth minimal surface. If $\eta \subset B_r(p) \cap \Sigma$ then η is null homologous in $B_r(p) \cap \Sigma$. In particular, the intersection of any minimal disk with a euclidean ball is still a disk.*

There is also an intrinsic version of the above. Indeed, for any $\Sigma \subset \mathbb{R}^3$ the Gauss-Codazzi equations give that

$$0 = R^\Sigma + |A^\Sigma|^2 - (H^\Sigma)^2$$

where here R^Σ is the scalar curvature of Σ. In other words, if Σ is minimal then

$$K^\Sigma = -|A^\Sigma|^2 \leq 0$$

where here K^Σ is the Gauss curvature of Σ which in dimension 2 agrees with the scalar curvature. In particular, any minimal surface in $\mathbb{R}^3$ has non-positive curvature. This implies that the exponential map of the surface has no conjugate points. In particular,

LEMMA 2.7. *Let Σ be a smooth minimal disk in $\mathbb{R}^3$. If $B_r^\Sigma(p) \subset \Sigma \backslash \partial\Sigma$ then $B_r^\Sigma(p)$ is a smooth region in Σ which is topologically a disk.*

A very important property of minimal surfaces in Euclidean space is the so-called monotonicity formula:

THEOREM 2.8. *Let Σ be a minimal surface with $x_0 \in \mathbb{R}^3$ then for all $0 < s < t$ so that $\partial\Sigma \cap B_t = \emptyset$ one has*

$$t^{-2} Area(B_t \cap \Sigma) - s^{-2} Area(B_s \cap \Sigma) = \int_{(B_t \backslash B_s)\cap\Sigma} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^4}$$

PROOF. The proof relies on two facts: First, the coarea formula (see [**?**]) which gives that for any proper Lipschitz map $h : \Sigma \to \mathbb{R}$ and any locally integrable function f and $t \in \mathbb{R}$

$$\int_{h \leq t} f |\nabla^\Sigma h| = \int_{-\infty}^t \int_{h=\tau} f \, d\tau \tag{4.1}$$

Second, the minimality Σ gives that

$$\Delta^\Sigma |x - x_0|^2 = 2\text{div}^{\,\Sigma}(x - x_0) = 4. \tag{4.2}$$

By Stokes' theorem (4.2) gives

$$4 Area(B_s \cap \Sigma) = 2 \int_{\partial B_s \cap \Sigma} |(x-x_0)^\top|$$

Combining this with (4.1) gives

$$\begin{aligned} \frac{d}{ds}\left(s^{-2} Area(B_s \cap \Sigma)\right) &= -2s^{-3} Area(B_s \cap \Sigma) + s^{-2} \int_{\partial B_s \cap \Sigma} \frac{|x-x_0|}{|(x-x_0)^\top|} \\ &= s^{-3} \int_{\partial B_s \cap \Sigma} \left(\frac{|x-x_0|^2}{|(x-x_0)^\top|} - |(x-x_0)^\top| \right) \\ &= s^{-3} \int_{\partial B_s \cap \Sigma} \frac{|(x-x_0)^N|^2}{|(x-x_0)^\top|} \end{aligned}$$

Integrating this and applying (4.1) finishes the proof. □

The real power of this formula comes when one considers weak notions of minimal surface. Nevertheless, for smooth minimal surfaces one useful consequence is a uniform lower bound on the area of the surface in a ball centered at a point on the surface. Hence, when considering sequences of smooth minimal surfaces it (along with the convex hull property) allows one to use a uniform mass bound in a ball to conclude a uniform bound on the number of components of the surface in half the ball.

The monotonicity formula can be generalized to a weighted version which will allow us to prove a useful mean-value property for . If Σ is a minimal surface and f a C^2 function on Σ then

$$t^{-k}\int_{B_t\cap\Sigma} f - s^{-k}\int_{B_s\cap\Sigma} f = \int_{(B_t\setminus B_s)\cap\Sigma} f\frac{|(x-x_0)^{\perp}|^2}{|x-x_0|^4} \tag{4.3}$$

$$+\frac{1}{2}\int_s^t \tau^{-3}\int_{B_\tau\cap\Sigma}(\tau^2-|x-x_0|^2)\Delta^{\Sigma} f d\tau \tag{4.4}$$

which is proved in the same manner as the preceding result. This formula is particularly useful in giving a mean-value type inequality.

PROPOSITION 2.9. *Suppose that $\Sigma\subset\mathbb{R}^3$ is a smooth minimal surface $x_0\in\Sigma$ and $s>0$ satifies $B_s(x_0)\cap\partial\Sigma=\emptyset$. If f is a non-negative function on Σ with*

$$\Delta^{\Sigma} f \geq -\lambda s^{-2} f \tag{4.5}$$

then

$$f(x_0)\leq e^{\lambda/2}\frac{\int_{B_s\cap\Sigma} f}{\pi s^2} \tag{4.6}$$

PROOF. Set

$$g(r)=r^{-2}\int_{B_r\cap\Sigma} f$$

then the weighted monotonicity formula implies that

$$g'(r)\geq -\frac{\lambda}{2}s^{-2}r^{-1}\int_{B_r\cap\Sigma} f = -\frac{\lambda}{2}s^{-2}rg(r)$$

where $s>r>0$ is fixed. In other words,

$$\frac{d}{dr}\ln g(r)\geq -\frac{\lambda}{2}s^{-2}r\geq -\frac{\lambda}{2s},$$

Hence,

$$e^{\frac{\lambda r}{2s}}g(r)$$

is non-decreasing in r and achieves its maximum at $r=t$. The smoothness of Σ and continuity of f ensure that

$$\lim_{r\to 0} e^{\frac{\lambda r}{2s}}g(r)=\pi f(x_0)$$

which proves the proposition. $\square$

5. Minimal Graphs

We have already seen that any surface Σ in euclidean space is near a point $p\in\Sigma$ modelled on the graph over $T_p\Sigma$. of a function u. Moreover, a uniform curvature bound gives a uniform scale about p on which Σ may be written as a graph in this way. This allows one to make contact with standard facts about real valued functions. For this reason it is useful to study *minimal graphs* that is minimal surfaces that can be expressed as the graph of a function.

To that end suppose $\Omega \subset \mathbb{R}^2$ is an open region and $u : \Omega \to \mathbb{R}$ is a C^2 function. Then as we saw

$$H^{\Sigma}(p) = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

where p is identified with a point in Ω in the obvious way. In other words, if Σ is minimal if and only if u satisfies the *minimal surface equation*:

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{5.1}$$

Notice that as long as $|\nabla u|$ is uniformly bounded in Ω then u satisfies a uniformly elliptic second order linear equation that is in divergence form. The theory of such equations will be taken for granted (see for instance [**?, ?**]). In particular, the C^2 assumption is (more than) enough to ensure that u is C^∞ (in fact real analytic), moreover one gets standard elliptic estimates (for instance that the C^0 norm of u controls higher derivatives). Minimal graphs in $\mathbb{R}^3$ turn out to have more rigidity then this – namely a priori control on the second derivative. This is most strikingly captured by the classical Bernstein theorem

Theorem 2.10. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function satisfying* (5.1)*. Then u is an affine function, i.e. the graph of u is a plane.*

Remark 2.11. This holds for all $n \leq 7$ by work of De Giorgi, Almgren and Simons. the theorem fails for $n \geq 8$ by a construction of Bombieri, De Giorgi and Giusti.

One approach to the proof of this theorem is to use stability properties of minimal graphs. For that reason we defer the proof until the next section wherein we will discuss such properties. As alluded to, all minimal graphs are stable – in fact – they have even stronger variational properties:

Proposition 2.12. *Let $u : \Omega \to \mathbb{R}$ be solution to* (5.1) *C^∞ up to $\partial\Omega$. Then $\Sigma = \Gamma_u$ is area minimizing with in $\Omega \times \mathbb{R}$ in other words, if Σ' is a surface in $\Omega \times \mathbb{R}$ with $\partial\Sigma = \partial\Gamma_u$ then $Area(\Sigma') \geq Area(\Sigma)$. Moreover, if Ω is convex then Σ is area minimizing in all of $\mathbb{R}^3$. In other words if Σ' is any surface with $\partial\Sigma' = \partial\Sigma$ then $Area(\Sigma') \geq Area(\Sigma)$.*

Proof. We note that if Ω is convex then the nearest point projection map $\Pi : \mathbb{R}^3 \to \Omega \times \mathbb{R}$ is a distance non-increasing Lipschitz map which is identity on $\Omega \times \mathbb{R}$ so $Area(\Pi(\Sigma')) \leq Area(\Pi(\Sigma))$ and we are in the first case.

We conclude the proof by noting that vertical translates of Σ give a foliation of $\Omega \times \mathbb{R}$. This allows us to extend $\mathbf{n}$ the normal to Σ to a vector field on all of $\Omega \times \mathbb{R}$ in a natural way. Let ω be the two form given by

$$\omega(X, Y) = \det(X, Y, \mathbf{n})$$

One can verify that $d\omega = 0$ and that $|\omega(X, Y)| \leq 1$ for X, Y an orthonormal set of vectors with equality only when both X and Y are tangent to (a translate of) Σ. The result then follows by Stokes' theorem, namely

$$Area(\Sigma) = \int_\Sigma \omega = \int_{\Sigma'} \omega \leq Area(\Sigma').$$

□

Remark 2.13. We remark that such a calibration can always be constructed from a minimal foliation.

We conclude with a simple application of the strong maximum principal.

Proposition 2.14. *Suppose that* $u, v : \Omega \to \mathbb{R}$ *satisfy* (5.1) *for a connected region* Ω *containing* 0 *and that* $u(0) = v(0) = 0$ *and* $\nabla u(0) = \nabla v(0) = 0$*. If* $u \geq v$ *in* Ω *then* u *and* v *are identically equal in* Ω*.*

Proof. we note that our assumptions on u and v at 0 imply that for any $\epsilon > 0$ there is a $\delta > 0$ so that $B_\delta(0) \subset \Omega$ and $|\nabla u|, |\nabla v| \leq \epsilon$ in $B_\delta(0)$. Consider the function

$$w = u - v \geq 0.$$

We verify that for ϵ sufficiently small then in the corresponding $B_\delta(0)$, w satisfies an equation of the form

$$\text{div } (A \nabla w) = 0$$

where A is the 2×2 matrix given by

$$A_{ij} = \frac{\delta_{ij}}{\sqrt{1 + |\nabla u|^2}} - \frac{\partial_i (u + v) \partial_j v}{(\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla v|^2})\sqrt{1 + |\nabla u|^2}\sqrt{1 + |\nabla v|^2}}$$

which can be verified to be uniformly positive definite when ϵ is small enough. As $w(0) = 0$ is a local minimum of w and w satisfies a uniformly elliptic equation, the strong maximum principle implies that w identically vanishes in $B_\delta(0)$. An open and closed argument then proves the claim. □

Remark 2.15. A simple consequence of this is that two smooth minimal surfaces Σ_1 and Σ_2 cannot touch at a single point.

6. Stability

We will now discuss in more detail properties of stable minimal surfaces $\mathbb{R}^3$. Many of the results will have local analogs in arbitrary three-manifolds. As we will see, the properties of such stable minimal surfaces will be crucial in obtaining global information.

Recall, given a minimal surface Σ we saw that the second variation of the minimal surface was closely related to the spectral properties of the stability operator

$$L = \Delta^\Sigma + |A^\Sigma|^2 + Ric(\mathbf{n}, \mathbf{n}).$$

We first recall some properties of linear elliptic operators. First of all for a fixed smooth domain Ω which is compact recall the following variational characterization of the first dirichlet eigenvalue of L on Ω:

$$\lambda_1(\Omega) = \inf \left\{ -\int_\Omega \eta L \eta | \eta \in C_0^\infty(\Omega), \int_\Omega \eta^2 = 1 \right\}.$$

Indeed, standard compactness properties of Sobolev spaces imply that there is a function $u \in H_0^1(\Omega)$ so that

$$\int_\Omega u^2 = 1$$

and

$$-\int_\Omega u L u = \lambda_1.$$

That is u is a weak solution to $Lu = \lambda_1 u$. The regularity theory for elliptic operators then gives that $u \in C^\infty(\Omega)$ with $u|_{\partial\Omega} = 0$, i.e. u is an eigenfunction associated to λ_1. By using the variational characterization and further properties of elliptic functions (i.e. the Harnack inequality) we see that we may take u so that it is strictly positive in Ω. As all eigenfunctions are orthogonal, this implies that there is only one eigenfunction associated to λ_1 and moreover any eigenfunction that is positive must be (up to a constant) u. In particular, if $u \geq 0$ is a positive solution to $Lu = 0$ vanishing on $\partial\Omega$ then this immediately implies that Ω is a stable region in Σ. The assumption that u vanish on $\partial\Omega$ is not needed:

LEMMA 2.16. *(Fischer-Colbrie and Schoen [**?**]) Let Σ be an oriented minimal surfaces in M with L its stability operator and $\Omega \subset \Sigma$ a bounded smooth domain. If there is a positive function u on Ω with $Lu = 0$ the Ω is stable.*

PROOF. Let $V = |A^\Sigma|^2 + Ric(\mathbf{n}, \mathbf{n})$ so $L = \Delta^\Sigma + V$ is the stability operator. Since $u > 0$ on Ω one has that $w = \log u$ is well-defined and satisfies

$$\Delta^\Sigma w = \frac{\Delta^\Sigma u}{u} - \frac{|\nabla^\Sigma u|^2}{u^2} = -V - |\nabla^\Sigma w|^2. \tag{6.1}$$

Let $\eta \in C_0^\infty(\Omega)$ be a smooth function of compact support. Multiplying (6.1) by η^2 and integrating gives

$$\begin{aligned}\int_\Omega \eta^2 V + \int_\Omega \eta^2 |\nabla^\Sigma w|^2 &= -\int_\Omega \eta^2 \Delta^\Sigma w \\ &\leq \int_\Omega |\eta||\nabla^\Sigma \eta||\nabla^\Sigma w| \\ &\leq \int_\Omega \eta^2 |\nabla^\Sigma w|^2 + \int_\Omega |\nabla^\Sigma \eta|^2\end{aligned}$$

were the first inequality followed by integrating by parts and the Cauchy-Schwarz inequality and the second from the absorbing inequality. Hence,

$$\int_\Omega (|A^\Sigma|^2 + Ric(\mathbf{n}, \mathbf{n})\eta^2 \leq \int_\Omega \eta^2.$$

As η is arbitrary we conclude that $\lambda_1(\Omega) \geq 0$. □

We refer to solutions to

$$Lu = 0$$

as *Jacobi functions*. One can verify that the stability operator L is the linearization of the minimal surface equation and hence Jacobi functions should correspond to infinitesimal deformations through minimal surfaces. A consequence of the above, if Σ admits a positive Jacobi function then Σ is stable.

We present some applications of this: First of all, if Σ in M is a stable minimal surface satisfying our standing assumptions and $\Pi : \hat{M} \to M$ is a Riemannian covering map then $\hat{\Sigma} = \Pi^{-1}(\Sigma)$ is still a stable minimal surface. Indeed, if u is a positive solution on Σ to $Lu = 0$ then $u \circ \Pi$ positive Jacobi function on $\hat{\Sigma}$. Such a result would not be true with out our assumption that Σ was oriented. For instance the unoriented minimal surface $\mathbb{RP}^2 \subset \mathbb{RP}^3$ can be checked to be stable and to be covered by $\mathbb{S}^2 \subset \mathbb{S}^3$, which is not stable. More generally, the existence of positive Jacobi functions on a minimal surface can often be shown for purely geometric reasons. For instance, suppose $\Sigma = \Gamma_u \subset \mathbb{R}^3$ is a minimal graph over the x_1-x_2 plane. We've already seen that such surfaces are area-minimizing (in an appropriate sense) and so are stable, but can also see this by noting that that vertical translation of the Σ is a variation of Σ which gives rise to the Jacobi function

$$g(\mathbf{e}_3, \mathbf{n}) = \frac{1}{\sqrt{1+|\nabla u|^2}} \tag{6.2}$$

which is manifestly positive.

We now want to discuss some of the geometric consequences of stability. A natural starting point is to try and understand complete stable minimal surfaces in $\mathbb{R}^3$. Let us say a complete surface Σ has *(intrsinsic) quadratic area growth* if for some $p \in \Sigma$ there is a constant $C > 0$ so that

$$Area(\mathcal{B}^\Sigma_r(p) \leq Cr^2. \tag{6.3}$$

For instance, as $\mathcal{B}^\Sigma_r(p) \subset \Sigma_{r,p}$, if $\Sigma = \Gamma_u$ is an entire graph then by using comparisions with spherical caps and the area-minimizing property of graphs one concludes that Σ has quadratic area growth. Surfaces with quadratic area growth admit a good sequence of test functions:

Proposition 2.17. *Let Σ be a surface with (intrinsic) quadratic area growth then there is a family of test functions $\phi_i \in C^\infty_0(\Sigma)$ satisfying*

(1) $0 \leq \phi_i \leq 1$
(2) $\phi_i = 1$ *on* $\mathcal{B}^\Sigma_{r_i}(p)$ *for some fixed* $p \in \Sigma$ *with* $r_i \to \infty$.
(3) $\lim_{i\to\infty} \int_\Sigma |\nabla^\Sigma \phi_i|^2 \to 0$.

Remark 2.18. The existence of such sequence of cut-offs is equivalent to saying that the surface is *parabolic* – that is it admits non-constant superharmonic functions. This is a property of the conformal structure of Σ.

Proof. The existence of such cut-offs follows from the "logarithmic cut-off trick". Namely, define ν_R a Lipschitz function on $\mathbb{R}^{\geq 0}$ by

$$\nu_R(r) = \begin{cases} 1 & 0 \leq r < R \\ 2 - 2\frac{\log r}{\log R} & R < r^2 < R^2 \\ 0 & r \geq R^2 \end{cases}.$$

Now fix a $p \in \Sigma$ and a sequence of $r_i \to \infty$ and set $r(x) = d^\Sigma(p,x)$ and

$$\phi_i(x) = \nu_{r_i}(r(x))$$

By the chain rule and the fact that the distance function has Lipschitz norm 1 we verify that

$$|\nabla^\Sigma \phi_i| \leq \frac{2}{r\log r_i}\chi_{\sqrt{r_i},r_i}$$

where $\chi_{\sqrt{r_i},r_i}$ is the characteristic function of $\mathcal{B}^\Sigma_{r_i}(p)\backslash\mathcal{B}^\Sigma_{\sqrt{r_i}}(p)$ We then can compute that

$$\begin{aligned}\int_\Sigma |\nabla^\Sigma\phi_i|^2 &= \int_{\mathcal{B}^\Sigma_{r_i}(p)} |\nabla^\Sigma \phi_i|^2 \\ &\leq \frac{4}{(\log r_i)^2}\sum_{l=\frac{1}{2}\log r_i}^{\log r_i} \int_{\mathcal{B}^\Sigma_{e^l}\backslash\mathcal{B}^\Sigma_{e^{l-1}}} r^{-2} \\ &\leq \frac{4}{(\log r_i)^2}\sum_{l=\frac{1}{2}\log r_i}^{\log r_i} Ce^2.\end{aligned}$$

The last inequality uses the quadratic area growth. Taking the sum we obtain

$$\int_\Sigma |\nabla^\Sigma \phi_i|^2 \leq \frac{2Ce^2}{\log r_i}$$

and the left hand side tends to 0 as $r_i \to \infty$. □

Using the existence of these test functions we can show the following rigidity result for stable minimal surfaces

PROPOSITION 2.19. *Let Σ be a complete, stable minimal surface in $\mathbb{R}^3$ that is oriented and has quadratic area growth then Σ is a plane.*

PROOF. As Σ has quadratic area growth we may choose cut-offs ϕ_i as in the preceding Proposition. Since Σ is stable the stability inequality gives that

$$\int_{\mathcal{B}^\Sigma_{\sqrt{r_i}}} |A^\Sigma|^2 \leq \int_\Sigma |A|^2 \phi_i^2 \leq \int_\Sigma |\nabla^\Sigma \phi_i|^2$$

Hence for any fixed $R > 0$ by taking $r_i \to \infty$ we have that

$$\int_{\mathcal{B}^\Sigma_R} |A^\Sigma|^2 = 0$$

i.e. Σ is totally geodesic and hence a plane. □

In particular, we have proved the Bernstein theorem in $\mathbb{R}^3$.

We will now show that for stable surfaces in $\mathbb{R}^3$ the assumption of quadratic area growth is unnecessary. Our proof will use the Gauss-Bonnet formula and as such is highly dependent of Σ being a surface. It is unknown whether such rigidity result with no a priori bounds holds in higher dimensions.

THEOREM 2.20. *Let Σ be a complete, oriented stable minimal surface in $\mathbb{R}^3$. Then Σ is a plane.*

PROOF. As we have seen, the universal cover of a stable oriented surface is still stable. In particular, we may pass to the universal cover and assume that Σ is topologically a disk. In particular, for a fixed $p \in \Sigma$ if we define $r(x) = d^\Sigma(x, p)$ then, as $K^\Sigma = -\frac{1}{2}|A^\Sigma|^2 \leq 0$, away from p, r is a smooth function and hence $\mathcal{B}^\Sigma_r(p)$ is a smooth domain. Notice that away from p, $|\nabla^\Sigma r| = 1$. By the first variation formula and the Gauss-Bonnet formula if we set $\ell(r) = Length(\partial \mathcal{B}^\Sigma_r(p))$ then

$$\begin{aligned}
\ell'(r) &= \int_{\mathcal{B}^\Sigma_r} \kappa_g \\
&= 2\pi\chi(\mathcal{B}^\Sigma_r) - \int_{\mathcal{B}^\Sigma} K^\Sigma d\mu \\
&= 2\pi + \frac{1}{2}\int_{\mathcal{B}^\Sigma} |A^\Sigma|^2 d\mu
\end{aligned}$$

Now let $\eta \in C^\infty_0(\mathbb{R}^{\geq 0})$ satisfy $\eta' \leq 0$ and $\eta(r_0) = 0$ and let $f(x) = \eta(r(x))$ be a cut-off in $\mathcal{B}_{r_0}$. As Σ is stable we have by the Co-area formula that

$$\begin{aligned}
0 \leq -\int_{\mathcal{B}_{r_0}} fLf &= \int_{\mathcal{B}_{r_0}} |\nabla^\Sigma f|^2 - \int_{\mathcal{B}_{r_0}} |A|^2\eta^2 \\
&= \int_0^{r_0} (\eta'(s))^2 \ell(s) ds - \frac{1}{2}\int_{\mathcal{B}_{r_0}} |A|^2 f - \frac{1}{2}\int_0^{r_0} \eta(s)^2 \int_{\partial B_s} |A|^2 ds
\end{aligned}$$

Integrating by parts, and substituting in our computation involving the Gauss-Bonnet theorem we obtain

$$\begin{aligned} 0 \leq \frac{1}{2}\int_{\mathcal{B}_{r_0}} |A|^2 f^2 &\leq \int_0^{r_0} (\eta'(s))^2 \ell(s) ds - \frac{1}{2}\int_0^{r_0} \eta(s)^2 \frac{d}{ds}\int_{B_s} |A|^2 ds \\ &= \int_0^{r_0} (\eta'(s))^2 \ell(s) ds + \frac{1}{2}\int_0^{r_0} (\eta(s)^2)' \int_{B_s} |A|^2 ds \\ &= \int_0^{r_0} (\eta'(s))^2 \ell(s) ds - \int_0^{r_0} (\eta(s)^2)'(2\pi - \ell'(s)) ds \end{aligned}$$

If we take $\eta(s) = 1 - s/r_0$ then this gives

$$\frac{1}{2}\int_{\mathcal{B}_{r_0}} |A|^2 f^2 + 2r_0^{-1}\int_0^{r_0} (1 - \frac{s}{r_0})\ell'(s) ds \leq r_0^{-2}\int_0^{r_0} \ell(s) + 2\pi$$

Integrating by parts again and then using the co-area formula then gives

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{B}_{r_0}} (1 - \frac{r}{r_0})^2 |A|^2 + 2r_0^{-2}\int_0^{r_0} \ell(s) ds &\leq r_0^{-2}\int_0^{r_0} \ell(s) + 2\pi \\ \frac{1}{2}\int_{\mathcal{B}_{r_0}} (1 - \frac{r}{r_0})^2 |A|^2 + 2r_0^{-2} Area(\mathcal{B}_s) &\leq r_0^{-2} Area(\mathcal{B}_s) + 2\pi \end{aligned}$$

Hence

$$\frac{1}{2}\int_{\mathcal{B}_{r_0}} (1 - \frac{r}{r_0})^2 |A|^2 + r_0^{-2} Area(\mathcal{B}_s) \leq 2\pi.$$

Hence any stable disk has quadratic area growth and we can appeal to the previous results.

□

It will be highly useful to have an *effective* version of this result. In other words we would like a result that tells us that if we have a minimal surface Σ and $\mathcal{B}_r^{\Sigma}(p) \subset \Sigma \backslash \partial\Sigma$ was a large stable disk, then Σ would need to be very flat near p. Moreover, it will also be useful to get a quantitative estimate on this flatness. The above arguments can readily be seen to give an L^2 estimate on the flatness, however, in practice we want to have a pointwise estimate. This is because then we can appeal to Lemma 2.1 and write our surface as a graph on fixed scale, which, as we have discussed, allows access to the usual theory of elliptic PDE. By considering rescalings of the catenoid, we see that we can not directly control the L^∞ norm of $|A|$ by the L^2 norm of $|A|$. Indeed, on a fixed ball a sequence of such examples can have uniform bound on the total curvature, but with the second fundamental form blowing up at the neck. Notice in this example the the total curvature near the neck is always approximately 4π. In other words, the total curvature is concentrating.

The estimate for stable minimal surfaces in $\mathbb{R}^3$ is as follows:

THEOREM 2.21. *(Schoen [**?**]) Let Σ be a compact, oriented and stable minimal surface in $\mathbb{R}^3$. Fix a point $p \in \Sigma$ and radius $r_0 > 0$ so that $\mathcal{B}_{r_0}^{\Sigma}(p) \subset \Sigma \backslash \partial\Sigma$. Then there is a universal $C > 0$ so that for all $0 < \sigma \leq r_0$ one has the scale invariant* *estimate*

$$\sup_{\mathcal{B}^{\Sigma}_{r_0-\sigma}} |A^{\Sigma}|^2 \leq C\sigma^{-2}$$

7. Proof of Schoen's Estimate

Our method for obtaining such L^{∞} bounds will involve a variant on the Bochner technique, applied to the second fundamental form. The key observation, is the following inequality attributed to Simons:

LEMMA 2.22. *Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be a minimal hyper-surface then*

$$\Delta^{\Sigma}|A^{\Sigma}|^2 \geq -2|A^{\Sigma}|^4 + 2\left(1 + \frac{2}{n-1}\right)|\nabla_{\Sigma}|A||^2 \tag{7.1}$$

REMARK 2.23. When $n = 3$ this inequality actually takes the form of an equality

$$\Delta^{\Sigma}|A^{\Sigma}|^2 = -2|A^{\Sigma}|^4 + 4|\nabla^{\Sigma}|A||^2$$

PROOF. The proof is somewhat involved though relatively straightforward, so in the interest of time I will omit most of the details (see [**1**] for complete details). I will illustrate one part of the proof that is of independent interest. By computing along the same lines as in the Bochner formula one obtains the following *Simons' equation*:

$$\Delta^{\Sigma}|A^{\Sigma}|^2 = -2|A^{\Sigma}|^4 + 2|\nabla^{\Sigma}A^{\Sigma}|^2$$

where here $\nabla^{\Sigma}A^{\Sigma}$ is the covariant derivative of A^{Σ}. A weaker form of (7.1) would then follows immediately Kato's inequality. To see how to obtain the full (7.1) we use the fact that the trace of A^{Σ} is 0. First fix a point $p \in \Sigma$ and choose $E_1, \ldots, E_{n-1}$ an orthonormal frame on Σ so that at p the E_i diagonalize A^{Σ} (recall the second fundamental form is symmetric). Hence by the Cauchy-Schwarz inequality if we set $a_{ij} = A^{\Sigma}(E_i, E_j)$:

$$\begin{aligned} 4|A|^2|\nabla^{\Sigma}|A||^2 &= |\nabla^{\Sigma}|A|^2| \\ &= \sum_{k=1}^{n-1}\left(\sum_{i,j=1}^{n-1}(a_{ij}^2)_k\right)^2 \\ &\leq 4\sum_{k=1}^{n-1}\left(\sum_{i=1}^{n-1}a_{ii,k}a_{ii}\right)^2 \\ &\leq 4|A|^2\sum_{k=1}^{n-1}a_{ii,k}^2 \end{aligned}$$

note we used that at p $a_{ij} = 0$ for $i \neq j$. Hence

$$|\nabla^{\Sigma}|A||^2 \leq \sum_{k=1}^{n-1} a_{ii,k}^2.$$

Now by the minimality of Σ:

$$\begin{aligned}\sum_{k=1}^{n-1} a_{ii,k}^2 &= \sum_{i\neq k} a_{ii,k}^2 + \sum_{i=1}^{n-1} a_{ii,i}^2 \\ &= \sum_{i\neq k} a_{ii,k}^2 + \sum_{i=1}^{n-1}\left(\sum_{i\neq j} a_{jj,i}\right)^2 \\ &\leq \sum_{i\neq k} a_{ii,k}^2 + (n-2)\sum_{i=1}^{n-1}\sum_{i\neq j} a_{jj,i}^2 \\ &= (n-1)\sum_{i\neq k} a_{ik,i}^2 \\ &= \frac{n-1}{2}\left(\sum_{i\neq k} a_{ik,i}^2 + \sum_{i\neq k} a_{ki,i}^2\right)\end{aligned}$$

where we also use the Codazzi equations. Hence

$$\begin{aligned}\left(1+\frac{2}{n-1}\right)|\nabla^\Sigma |A^\Sigma|^2|^2 &\leq \sum_{i,k=1}^{n-1} a_{ii,k}^2 + \sum_{i\neq k} a_{ik,i}^2 + \sum_{i\neq k} a_{ki,i}^2 \\ &\leq \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 \\ &= |\nabla^\Sigma A^\Sigma|^2.\end{aligned}$$

□

Remark 2.24. For our purposes Simons' equation (7) will suffice. The full strength of the inequality is needed for the higher dimensional regularity theory.

Our main use of this is to show that a uniform bound on $|A|$ allows us to control $|A|^2$ by the average curvature

Corollary 2.25. *Let Σ be a minimal surface in $\mathbb{R}^3$. Suppose that $p \in \Sigma$ and $B_r(p) \cap \partial\Sigma = \emptyset$. If $\sup_{\Sigma_{p,r}} |A|^2 \leq Cr^{-2}$ then*

$$|A|(p)^2 \leq \frac{e^{C/2}}{\pi r^2}\int_{\Sigma_{p,r}} |A|^2$$

Proof. By the Simons' inequality we have

$$\Delta^\Sigma |A|^2 \geq -r^{-2}C|A|^2$$

Hence by the mean value inequality we have

$$|A|^2(p) \leq \frac{e^{C/2}}{\pi r^2}\int_{\Sigma_{p,r}} |A|^2.$$

□

We will now show that when the total curvature of a minimal surface is sufficiently small then the total curvature controls the L^∞ norm of $|A^\Sigma|$. The area estimates and stability inequality we've seen above can then be used to prove the

Theorem 2.21. In particular, we will show the following result originally due to Choi and Schoen:

THEOREM 2.26. *There exists an* $\epsilon > 0$ *so that if:* Σ *is a compact minimal surface in* $\mathbb{R}^3$ *and* $0 < \delta \leq 1$ *is fixed then if* $\mathcal{B}_{r_0}(x) \subset \Sigma \backslash \partial\Sigma$ *and*

$$\int_{\mathcal{B}_{r_0}(x)} |A^\Sigma|^2 \leq \epsilon \delta^2$$

then for all $0 < \sigma \leq r_0$ *if* $y \in \mathcal{B}^\Sigma_{r_0 - \sigma}(x)$ *then*

$$\sigma^2 |A^\Sigma|^2(y) \leq \delta^2.$$

The key ingredients of the proof will be

(1) The monotonicity and scaling invariance of the total curvature,
(2) The mean value inequality and Simons' inequality (i.e. Corollary 2.25) and;
(3) A blow-up argument.

Let us discuss the last one first as it will appear in numerous guises:

DEFINITION 2.27. For a fixed $C > 0$ we say that a pair $(p, s) \in \Sigma \times \mathbb{R}^+$ is a (*intrinsic*) *C blow-up pair* if

$$\mathcal{B}^\Sigma_s(p) \subset \Sigma \backslash \partial\Sigma$$

and

$$\sup_{\mathcal{B}^\Sigma_s(0)} |A^\Sigma|^2 \leq 4|A^\Sigma|^2(p) = 4C^2 s^{-2}.$$

That is a C blow up pair consists of a point p and a scale s so that the scale s inversely proportional to the curvature at p and on the scale s the curvature at p is almost the maximum point for the curvature. A blow-up pair is scale invariant in that if (p, s) is a C blow-up pair then after a rescaling by s^{-1}, $(p, 1)$ is a blow-up pair in the new surface. Similarly, if (p, s) is a C blow-up pair then after a rescaling by Cs^{-s}, $\mathcal{B}^\Sigma_C(p)$ has uniform curvature bounded by 4 while $|A|(p) = 1$.

We have the following simple lemma which shows that when the curvature of a surface is sufficiently large one always can find a C blow-up pair.

LEMMA 2.28. *Fix* $C > 0$ *and Suppose* Σ *is a compact surface in* $\mathbb{R}^3$ *so that*

$$\mathcal{B}^\Sigma_{r_0}(p) \subset \Sigma \backslash \partial\Sigma$$

and

$$\sup_{\mathcal{B}^\Sigma_{r_0/2}} |A^\Sigma|^2 \geq 16 C^2 r_0^{-2}.$$

Then there is a point q *and scale* $s > 0$ *so that* $\mathcal{B}^\Sigma_s(q) \subset \mathcal{B}^\Sigma_{r_0}$ *and* (q, s) *is a* C *blow-up pair.*

PROOF. With $r(x) = d^\Sigma(x, p)$ define the function

$$F(x) = (r(x) - r_0)^2 |A|^2.$$

This is a Lipschitz function on $\mathcal{B}^\Sigma_{r_0}(p)$ that vanishes on $\partial\mathcal{B}^\Sigma_{r_0}$. Since F is continuous, non-negative and vanishes on $\partial\mathcal{B}^\Sigma(p)$ then F achieves its positive maximum somewhere in the interior of $\mathcal{B}^\Sigma_{r_0}(p)$. Let $q \in \mathcal{B}^\Sigma_{r_0}(p)$ be the point where F achieves its maximum. Fix a $\sigma > 0$ so that $q \in \partial\mathcal{B}^{\Sigma}_{r_0-\sigma}(p)$. From $\sup_{\mathcal{B}^{\Sigma}_{r_0/2}} |A^{\Sigma}|^2 \geq 16C^2 r_0^{-2}$ one verifies that

$$F(q) \geq 4C^2.$$

Now pick s so that

$$s^2|A|^2 = C^2.$$

As $\sigma^2|A|^2 \geq 4C^2$ this implies that $2s \leq \sigma$. On the other hand,

$$\sup_{\mathcal{B}^{\Sigma}_s(q)} \frac{\sigma^2}{4}|A|^2 \leq \sup_{\mathcal{B}^{\Sigma}_{\sigma/2}(q)} \frac{\sigma^2}{4}|A|^2 \leq \sup_{\mathcal{B}^{\Sigma}_{\sigma/2}(q)} F \leq \sigma^2|A|^2(q).$$

Where the last inequality holds as F achieves its maximum at q. □

REMARK 2.29. The significance of this lemma is to find the point at which the curvature has nearly the worst behavior, but that is also fairly far from the boundary. One hopes to understand the structure of the surface at these points and perhaps to rule them out altogether.

Let us now prove the Choi-Schoen theorem:

PROOF. Suppose there is a $\sigma > 0$ and a point $y \in \mathcal{B}_{r_0-\sigma}(x)$ so that

$$\sigma^2|A|^2(y) \geq \delta^2$$

By Lemma 2.28 there is then a point $p \in \mathcal{B}^{\Sigma}_{\sigma}(p)$ and a scale $s > 0$ so that (p, s) is a $\frac{\delta}{4}$ blow-up pair and $\mathcal{B}^{\Sigma}_s(p) \subset \mathcal{B}^{\Sigma}_{\sigma}(y)$. Rescaling Σ about p by $\frac{1}{s}$ we get an intrinsic ball $\mathcal{B}^{\Sigma}_1(p)$ on which $|A|^2 \leq \frac{\delta^2}{4} \leq 1$ and so that $|A|^2(p) = \frac{\delta^2}{16}$. Notice in particular, that by the graphical approximation lemma, there is some uniform $1 > \gamma > 0$ so that $\Sigma_{p,\gamma} \subset \mathcal{B}^{\Sigma}_1(p)$. By scaling invariance of total curvature and the monotonicity properties of the the total curvature we have that

$$\int_{\Sigma_{p,\gamma}} |A|^2 \leq \epsilon\delta^2.$$

As a consequence, by Corollary 2.25

$$\frac{\delta^2}{16} = |A|^2(p) \leq \frac{e^{\gamma^2/2}}{\pi\gamma^2}\epsilon\delta^2$$

In particular, if $\epsilon < \frac{1}{16}\left(\frac{e^{\gamma^2/2}}{\pi\gamma^2}\right)^{-1}$ we obtain a contradiction proving the theorem. □

The proof of Theorem 2.21 is now straightforward. For the sake of completeness, we include the details.

PROOF. As before, by passing to the universal cover, we may assume that Σ is a topological disk. Suppose that there was a point $p \in \Sigma$ so that $\mathcal{B}^{\Sigma}_{r_0}(p) \subset \Sigma\backslash\partial\Sigma$ while for some $0 < \sigma \leq r_0$:

$$\sup_{\mathcal{B}^{\Sigma}_{r_0-\sigma}} |A^{\Sigma}|^2 \geq 16C^2\sigma^{-2}$$

and $C > 0$ is large. If we let x be the point in $\mathcal{B}^{\Sigma}_{r_0-\sigma}$ so that

$$\sigma^2|A^{\Sigma}|^2(x) \geq 16C^2$$

Then we can find a C blow-up pair (y, s) so that $\mathcal{B}_s^\Sigma(y) \subset \mathcal{B}_\sigma^\Sigma(x) \subset \mathcal{B}_{r_0}^\Sigma(p)$. By rescaling by Cs^{-1} about y we obtain a stable region $\mathcal{B}_C^{\hat{\Sigma}} \subset \hat{\Sigma}$ so that $|A^{\hat{\Sigma}}|^2(y) = 1$. For C sufficiently large the proofs of Proposition 2.17 and Theorem 2.20 imply that one can take

$$\int_{\mathcal{B}_1^{\hat{\Sigma}}(y)} |A^{\hat{\Sigma}}|^2 < \epsilon$$

where were ϵ is given by Theorem 2.26. However, this implies that $|A^{\hat{\Sigma}}|^2(y) < 1$ which contradicts (y, C) being a C blow-up pair. □

CHAPTER 3

Choi-Schoen Theory

1. The Choi-Schoen Theorem

Let us begin by stating The following result of Choi and Schoen regarding the compactness properties of embedded minimal surfaces of a fixed genus in a three-manifold of positive Ricci curvature:

THEOREM 3.1. *Let (M, g) be a fixed compact 3-manifold so that $Ric > 0$. For each $l \geq 2$ the space of closed, embedded minimal surfaces in M of a fixed genus k is compact in the C^l topology. Indeed, for every integer $k \geq 0$ there is a constant $C = C(k)$ so that: If Σ is a closed, embedded minimal surfaces in M of a fixed genus k then:*

$$\sup_{\Sigma} |A^{\Sigma}| \leq C.$$

For simplicity we will assume that M is simply connected. If M is not simply connected, by Myers' theorem the universal cover $\hat{M}$ of M is compact and hence also satisfies the conditions of the theorem. All of conclusions descend to M – though the convergence is slightly trickier to define. When $\pi_1(M) = 0$ then any closed and connected embedded surface Σ divides M into two components and hence is oriented (just pick the outward normal of one of the regions).

Let us describe the C^l convergence of the theorem under the assumption that surfaces are oriented (which they are guaranteed to be when $\pi_1(M) = 0$). We will say that a sequence Σ_i of oriented closed embedded surfaces converges to Σ in the C^l topology (for $l \geq 2$) if the following holds: for any $\epsilon > 0$ there is an i_0 so that for $i > i_0$ there are functions u_i on Σ that satisfies

$$|u_i|_{C^l(\Sigma)} < \epsilon$$

so that Σ_i can be written as the normal exponential graph, Γ_{u_i}, of u_i over Σ. Here

$$\Gamma_{u_i} = \exp^M(u_i \mathbf{n})$$

where $\mathbf{n}$ is the normal vectorfield alogn Σ. We assume that $|u_i|_{C^0}$ is less then the focal radius of Σ. In particular, the convergence is with multiplicity one.

The proof of this theorem consists of three main steps. First we use properties of embedded minimal submanifolds in three-manifolds of positive Ricci curvature to get a priori area bounds and total curvature bounds. Second, we investigate the compactness theory given these a priori bounds and see that the convergence is smooth away from at most a finite number of points. Finally, the positivity of the Ricci curvature is used again to show that in fact the there are no singular points. This will also imply the curvature bound.

2. A priori Estimates

Let us begin by showing the needed a priori estimates. We first of all will need the following general result due to Yang and Yau which we present without proof

THEOREM 3.2. *(Yang-Yau [?]) Let Σ be a closed Riemannian surface of genus k. Then one has*

$$\lambda_1(\Sigma) \leq \frac{8\pi(1+k)}{Area(\Sigma)}$$

here

$$\lambda_1(\Sigma) = \inf\left\{\int_\Sigma |\nabla^\Sigma \eta|^2 d\mu | \eta \in C^\infty(\Sigma), \int_\Sigma \eta d\mu = 0\right\}$$

is the first Neumann eigenvalue of Δ^Σ.

The second fact we will need is the following result of Choi and Wang:

THEOREM 3.3. *(Choi-Wang [?]) Let M^n be an n-dimensional orientable Riemannian manifold with $Ric \geq \Lambda > 0$. If Σ^{n-1} is an embedded orienteable minimal hypersurface in M. Then*

$$\lambda_1(\Sigma) \geq \frac{\Lambda}{2}.$$

Combining these two facts we deduce

COROLLARY 3.4. *Let Σ be closed embedded minimal surface of genus k in a M a simply connected three-manifold with $Ric \geq \Lambda > 0$ and $|K_M| \leq K$ (here K_M is the sectional curvature of M). Then*

$$Area(\Sigma) \leq \frac{16\pi(1+k)}{\Lambda}$$

and

$$\int_\Sigma |A^\Sigma|^2 \leq \frac{32\pi(1+k)K}{\Lambda} + 4\pi(g-1).$$

PROOF. As M is simply connected it is automatically oriented, moreover as Σ is embedded it is also oriented. The area estimate then follows directly by combining the estimates of Yang-Yau and Choi-Wang. To show the total curvature bound, we note that the Gauss equations give (using the minimality of Σ) that

$$K^\Sigma = K^M - \frac{1}{2}|A^\Sigma|^2$$

Here K^M is the sectional curvature of M in the plane given by $T\Sigma$ Hence

$$|A^\Sigma|^2 \leq -2K^\Sigma + 2K^M \leq -2K^\Sigma + 2K$$

Integrating and using the Gauss-Bonnet theorem we obtain

$$\int_\Sigma |A^\Sigma|^2 = 4\pi(g-1) + 2K Area(\Sigma)$$

and so the total curvature bound follows from the area estimate. □

3. Compactness of Minimal submanifolds with a priori area and curvature bounds

We consider now a general compactness theory for minimal submanifolds with a priori bounds on the area and on the total curvature

PROPOSITION 3.5. *Let (M, g) be a fixed compact Riemannian manifold. Suppose that Σ_i is a sequence of embedded, oriented minimal surfaces in M with*

$$Area(\Sigma_i) \leq C_1 \tag{3.1}$$

and

$$\int_{\Sigma_i} |A^{\Sigma_i}|^2 \leq C_2. \tag{3.2}$$

Then up to passing to a subsequence, there is a smooth embedded minimal surface Σ in N and a finite set of points $\mathcal{S} = \{p_1, \dots, p_n\} \subset M$ so that:

1. *for each $l \geq 2$ each Σ_i converges in $C^l_{loc}(M \backslash \mathcal{S})$,*
2. *Σ_i converge to Σ in the Hausdorff topology*
3. *Σ satisfies (3.1) and (3.2).*

REMARK 3.6. We emphasize that the removability of the singularities (i.e. the fact that Σ is smooth) is really the key feature. We also point out that we will cheat a little in our proofs and use various results that we have stated and proved only in $\mathbb{R}^3$. The point is that everything we use will be taking place on very small balls in M on which (after rescaling) the metric looks essentially euclidean and as consequence the euclidean results (after a small modification) still hold. The details can be found in chapter 5 of [**1**].

In order to prove this we will need the following removable singularity result (as usual proved only in $\mathbb{R}^3$):

LEMMA 3.7. *Fix $C_1 > 0$ and let Σ be smooth embedded minimal surface in $B^1(0) \backslash \{0\} \subset \mathbb{R}^3$ so that $0 \in \bar{\Sigma}$. If*

$$Area(\Sigma \cap B_r) \leq C_1 r^2$$

and

$$\int_{\Sigma} |A^{\Sigma}|^2 \leq C_1.$$

then $B_{1/2}(0) \cap \bar{\Sigma}$ is a smooth embedded minimal surface.

REMARK 3.8. We take

$$Area(\Sigma \cap B_r) := \lim_{\rho \to 0} Area(\Sigma \cap B_r \backslash B_\rho)$$

and likewise for $\int_{\Sigma} |A^{\Sigma}|^2$

PROOF. We first claim that if, for some $r > 0$, $\bar{\Sigma} \cap B_r(0)$ is the graph of a C^1 function then the theorem is true. To see this suppose that $u : \Omega \to \mathbb{R}$ is a C^1 function so that $\Gamma_u = \bar{\Sigma} \cap B_r(0)$. Notice that

$$\sup_{\Omega} |u| + |\nabla u| \leq C$$

for some large constant C. Since u is smooth in $\Omega \backslash \{0\}$ and $|\nabla u|$ is bounded in Ω one can treat the minimal surface equation (5.1) as a uniformly elliptic equation with bounded coefficients. In particular, $u \in H^1(\Omega)$ is a weak solution to a uniformly elliptic equation with bounded coefficients in all of Ω. Standard elliptic theory then implies that $u \in H^2(\Omega)$ (see [**?**]) and and so $|\nabla u| \in H^1(\Omega)$. Standard elliptic theory can then be applied to show that u is smooth.

By the monotone convergence theorem we have

$$\lim_{r\to 0}\int_{B_r\cap\Sigma}|A^\Sigma|^2 \to 0$$

In particular, if $\epsilon > 0$ is given by Theorem 2.26 and $\delta_0 > 0$, is a fixed number to be determined we can take r_0 sufficeintly small so that

$$\int_{B_{r_0}\cap\Sigma}|A^\Sigma|^2 \leq \delta_0^2\epsilon.$$

By Theorem 2.26 if $r < r_0$ and $z \in B_r(0)\backslash B_{r/2}(0)$ we have

$$r^2|A^\Sigma|^2(z) \leq 4\delta_0^2.$$

We fix r for the moment and pick a $z_1 \in \partial B_{3/4r}(x)$. By slight modification of the proof of Lemma 2.1 $\Sigma_{z_1,r/4}$ is a graph over $T_{z_1}\Sigma$ of a function u which satisfies

$$|\nabla u| + r|\nabla^2 u| \leq C\delta_0.$$

(Recall in Lemma 2.1 we assumed only uniform curvature bound and got as a consequence a uniform bound on the graph– here we assume the curvature is small and get a corresponding smallness for the graph). By rotating $\mathbb{R}^3$ we assume that $\mathbf{n}(z_1) = \mathbf{e}_3$ and so take u a graph over the x_1-x_2 plane. We now repeat the argument with a point $z_2 \in \partial B_{3/4r}(x) \cap \partial B_{r/8}(z_1) \cap \Sigma_{z_1,r/4}$ (note the convex hull property ensures the existence of such a point). As before $\Sigma_{z_2,r/4}$ is a graph over $T_{z_2}\Sigma$ with small gradient. We see that $|\mathbf{n}(z_1) - \mathbf{n}(z_2)| \leq C\delta_0$. Continuing in this fashion we iteratively find $z_i \in \partial B_{3/4r}(x) \cap \partial B_{r/8}(z_{i-1}) \cap \Sigma_{z_{i-},r/4}$ so that $z_{i-2} \notin \Sigma_{z_i,r/8}$. Since each $\Sigma_{z_i,r/8}$ has approximately r^2 area and the total area of $(B_r\backslash B_{r/2}\cap)\Sigma$ is bounded by $C_1 r^2$ this procedure must stop after a finite number of steps (depending only on C_1). This corresponds to the component of $\partial B_{3/4r}\cap\Sigma$ containing z_1 closing up. By taking δ_0 sufficiently small (depending only on C_1) we see that the component of $(B_r(x)\backslash B_{r/2}(x))\cap\Sigma$ containing z_1 may be written as a graph of a function u over the x_1-x_2 plane so that

$$|\nabla u| + r|\nabla^2 u| \leq C\delta_0.$$

As a consequence of the above, each component of $\partial B_r \cap \Sigma$ has length bounded by Cr for some uniform constant C. That is, if we denote by Σ_i the components of $(B_{r_0}\backslash\{0\})\cap\Sigma$ then $\Sigma_i \cap \partial B_r$ has length bounded by Cr. Now since

$$\lim_{r\to 0}\int_{B_r}|A^\Sigma|^2 \to 0$$

we see by the Theorem 2.26 that we have for $r < r_0$ and any $z \in \partial B_r(0)\cap\Sigma$

$$r^2|A^\Sigma|^2(z) \leq \delta^2(r)$$

where $\delta_0 \geq \delta(r) \to 0$. In particular, by the length bound and the point-wise curvature estimates we will be able to continuously extend $\mathbf{n}$ from Σ_i to $\bar{\Sigma}_i$. Indeed, by a direct integration we see that

$$\sup_{z_1,z_2\in\partial B_r\cap\Sigma_i}|\mathbf{n}(z_1) - \mathbf{n}(z_2)| \leq C\delta(r).$$

In particular, when r is small we conclude that in $B_r \backslash B_{r/2}$ can be written as a very small graph over some plane. A priori such a plane can oscillate as $r \to 0$ which prevents us from extending $\mathbf{n}$ to $\bar{\Sigma}_i$. We claim that this is in fact not the case. To see this we fix one of the Σ_i and pick γ_r the curve $\partial B_{3r/4} \cap \Sigma_i$ which by the above has the property that $\frac{1}{r}\gamma_r$ converges to a circle of radius $3/4$ in some plane P_r through 0. In particular, we can solve the plateau problem with γ_r as boundary and obtain a minimal surface Γ_r with $\partial \Gamma_r = \gamma_r$. Since each γ_r is the graph over a convex domain in P_r a result of Rado [**?**] implies that Γ_r is also a graph over P_r. Consider the foliation given by translations Γ_r^t of Γ_r by an amount t in a direction orthogonal to P_r (such foliations can be constructed by other means [**?**] the strict maximum principle **??** implies that there is t_r so that $\Gamma_r^{t_r} \cap \bar{\Sigma}_i = \{0\}$. Indeed, by translating up or down we see that either $\Gamma_r = \bar{\Sigma}_i$ or there is a point of last contact between the family of Γ_r^t and $\overline{\Sigma}_i$. If we fix an r_0 small enough to make the above work and denote by $\mathbf{n}_0$ the normal to $\Gamma_{r_0}^{t_{r_0}}$ at 0 we claim that (up to reflection) that for $p \in \Sigma_i$

$$\lim_{p \to 0} \mathbf{n}(p) = \mathbf{n}_0.$$

This is seen by noting that $\Sigma_i \cap B_{3/4r_0}$ lies on one side of $\Gamma_{r_0}^{t_{r_0}}$. Then for r much smaller then r_0 if $p \in \gamma_r$ and $\mathbf{n}(r)$ is far from $\mathbf{n}_0$ the circle γ_r would have to cut through $\Gamma_{r_0}^{t_{r_0}}$.

Hence we see that each Σ_i can be extended to a C^1 surface across 0. By choosing r_0 sufficiently small we then see that each $B_r \cap \Sigma_i$ can we written as the graph of a C^1 function u_i. Hence, by our remarks at the beginning each Σ_i is a smooth minimal surfaces through 0. As Σ is embedded in $B_1 \backslash \{0\}$ we can order the components of $\Sigma_i \cap \left(B_{r_0} \backslash B_{r_0/2}\right)$ by height and this ordering is preserved in $B_{r_0} \backslash \{0\}$. Hence, distinct Σ_i and Σ_j can meet only at 0, but this would violate the strict maximum principal. Hence, there is only one such component. ☐

We can now show Proposition 3.5:

Proof. As each Σ_i is a closed set we can pass to a subsequence so that Σ_i converges to Σ in the Hausdorff sense. We will attempt to show that Σ is a smooth minimal surface and that (after possibly passing to further subsets) the convergence is as claimed. Now for each of the Σ_i let

$$\nu_i(U) = \int_{U \cap \Sigma_i} |A^{\Sigma_i}|^2 \leq C_2.$$

be a Radon measure on M associated to Σ_i. By standard compactness results for Radon measures we can pass to a subsequence so that $\nu_i \to \nu$ converges weakly to a Radon measure ν so that $\nu(M) \leq C_2$. If $\epsilon > 0$ is the constant of Theorem 2.26 then we define

$$\mathcal{S} = \{x \in M | \nu(\{x\}) \geq \epsilon\}$$

clearly $\mathcal{S}$ consists of at most $\frac{C_2}{\epsilon}$ points. For a point $x \in M \backslash \mathcal{S}$ since ν is a Radon measure it is Borel regular and hence there is a scale s so that $\nu(B_{10s}(x)) < \epsilon$. In particular, for i sufficiently large one has

$$\int_{\Sigma_i \cap B_{5s}(x)} |A^{\Sigma_i}|^2 < \epsilon.$$

Hence, by Theorem 2.26 if $\hat{\Sigma}_i$ is a component of $B_{5s}(x) \cap \Sigma_i$ then

$$25s^2 \sup_{\hat{\Sigma}_i} |A^{\hat{\Sigma}_i}|^2 \leq 1.$$

This allows us to argue as in our Geometric Arzela-Ascoli theorem **??**. In particular, by 2.1 for each $z \in B_s(x) \cap \Sigma_i$ the component $(\Sigma_i)_{z,s}$ can be written as the graph of a function u_i^z over $T_z\Sigma_i$ with $|\nabla u_i^z| + s|\nabla^2 u_i^z| \leq 2$. As each Σ_i is minimal this means that u_i^z satisfies a uniformly elliptic equation with Lipschitz coefficients and hence the usual elliptic estimates give uniform $C^{2,\alpha}$ bounds on $B_{s/2} \subset T_z\Sigma$– indeed we get uniform $C^{l,\alpha}$ bounds for any $l \geq 2$ fixed. Moreover, by the monotonicity formula, $\frac{\pi}{8}s^2$ is lower bound on the area of any $(\Sigma_i)_{z,s}$ that happens to meet $B_{s/2}(x)$. If $N(y)$ is the number of such components the uniform area bound then we explicitly have

$$\frac{\pi}{16}s^2 N(y) \leq C_1$$

and so $N(y)$ is bounded independent of y and i.

By a diagonalization argument and the Arzela-Ascoli theorem and passing to a subsequence we can then conclude that in $M \backslash \mathcal{S}$ the Σ_i converge to Σ in the C^l topology. In particular, since $l \geq 2$ and each Σ_i is minimal we see that Σ is minimal in $M \backslash \mathcal{S}$ and hence smooth. Furthermore, as each Σ_i is embedded Σ could fail to be embedded at a point $p \in M \backslash \mathcal{S}$ only if there were points $p_i, p_i' \in \Sigma_i$ so that $p_i, p_i' \to p$ and $\mathcal{B}_\gamma^{\Sigma_i}(p_i) \cap \mathcal{B}_\gamma^{\Sigma_i}(p_i') = \emptyset$ for some $\gamma > 0$. As this would violate the strict maximum principal we see that Σ is embedded in $M \backslash \mathcal{S}$. The weak convergence of L^1 and (3.2) implies that

$$\int_\Sigma |A^\Sigma|^2 \leq C_2.$$

For any $x \in \mathcal{S}$ fix a $1 > r_x > 0$ so that $B_{r_x}(x) \cap \mathcal{S} = \{x\}$. The C^2 convergence is ensures that for any $0 < r_1 < r_2 < r_x$

$$Area(\Sigma \cap (B_{r_2}(x) \backslash B_{r_1}(x))) = \lim_{i \to \infty} Area(\Sigma_i \cap (B_{r_2}(x) \backslash B_{r_1}(x)))$$

On the other hand, by the monotonicity formula, one has

$$Area(\Sigma_i \cap (B_{r_2}(x) \backslash B_{r_1}(x))) \leq C_1 r_2^2$$

in particular, by letting $r_1 \to 0$ we see that $\Sigma \cap B_{r_x}(x)$ satisfies (after a rescaling) the conditions of Lemma 3.7. Hence Σ is a smooth embedded surface in all of M. □

Remark 3.9. We remark that Σ_i converge to (an integer multiple of) Σ also in the sense of varifolds.

4. Smooth compactness

In order to conclude the proof of Theorem 3.1 we need the following smooth version of Allard's regularity theorem.

Theorem 3.10. *There is an $\epsilon > 0$ and $C > 0$ so that: If $\Sigma \subset \mathbb{R}^3$ be a smooth compact minimal surface in $\mathbb{R}^3$ and $p \in \Sigma$ satisfies $B_r(p) \cap \partial\Sigma = \emptyset$ and*

$$\frac{Area(B_r(p) \cap \Sigma)}{\pi r^2} \leq 1 + \epsilon$$

then

$$r^2 |A|^2(p) \leq C.$$

Remark 3.11. Allard originally proved this theorem in the context of stationary varifolds, i.e. generalized minimal surfaces. The proof is quite difficult in the generalized setting see for instance [?]. We present a simplified proof due to White [?] that holds for *smooth* minimal surfaces (see also Theorem 2.10 of [1] for a less general but more direct approach using the Gauss-Bonnet formula).

Proof. Suppose the theorem fails to hold. Then one has a sequence of counterexamples Σ_i. These Σ_i should be smooth compact minimal surfaces so that there are $p_i \in \Sigma_i$ and $r_i > 0$ so that $B_{r_i}(p_i) \cap \partial\Sigma_i = \emptyset$ and

$$\frac{Area(B_{r_i} \cap \Sigma)}{\pi r_i^2} \leq 1 + \epsilon_i \to 1$$

while

$$r_i^2 |A|^2(p_i) = 4C_i \to \infty.$$

Given such a sequence, by Lemma 2.28 there are a sequence of C_i blow-up pairs (q_i, s_i) in Σ_i. By rescaling about q_i by $C_i s_i^{-1}$ and translating q_i to 0 we obtain s sequence of surfaces $\hat{\Sigma}_i$ so that $(0, C_i)$ is a C_i blow-up pair. In particular, $\sup_{\mathcal{B}_1^{\hat{\Sigma}_i}(0)} |A^{\hat{\Sigma}_i}|^2 \leq 4 = 4|A^{\hat{\Sigma}_i}|(0)$. By Lemma **??** there is a uniform $\gamma > 0$ so that $(\hat{\Sigma}_i)_{0,2\gamma}$ can be written as the normal graph over $T_0\hat{\Sigma}_i$ of a function u_i satisfying uniform gradient and Hessian estimates. In particular, by the Arzela-Ascoli theorem, up to passing to a subsequence, we have that $(\hat{\Sigma}_i)_{0,\gamma}$ converging in C^2 to some smooth minimal surface Σ with $0 \in \Sigma$ and $\partial\Sigma \subset B_\gamma(0)$. By the C^2 convergence, $|A^\Sigma|^2(0) = 1$. On the other hand, by scaling invariance of the area ratio and the monotonicity formula we have that $Area((\hat{\Sigma}_i)_{0,\gamma}) \leq \pi(1+\epsilon_i)\gamma^2$. By the C^2 convergence this implies that

$$Area(\Sigma \cap B_\gamma) \leq \pi\gamma^2.$$

Since Σ is a minimal surface, by the monotonicity formula, this can only occur if Σ is a cone. However, since Σ is smooth it must then be the disk D_γ which contradicts $|A^\Sigma|^2(0) = 1$. □

Corollary 3.12. *Fix a smooth region $\Omega \subset \mathbb{R}^3$. Suppose that Σ_i is sequence of smooth properly immersed minimal surfaces in Ω and Σ a fixed smooth and properly embedded surface in Ω. That is $\bar{\Sigma}_i \backslash \Sigma_i \subset \partial\Omega$ and likewise for Σ. Suppose that for all $B_r(p) \subset \Omega$*

$$Area(\Sigma_i \cap B_r(p)) \to Area(\Sigma \cap B_r(p))$$

Then for any $l \geq 2$, Σ_i converges to Σ in $C^l_{loc}(\Omega)$.

As a consequence of Theorem 3.10, Theorem 3.1 will be proved provided that we can show that the convergence given by Proposition **??** is with multiplicity one. Heuristically, you should think of this as coming from the lack of stable oriented closed surfaces in M when M has positive Ricci curvature. The idea is that if the multiplicity was greater than one, one could construct a positive Jacobi function on Σ by taking the difference between two of the leaves collapsing to Σ (the positivity follows from the the fact that each Σ_i is embedded). There is a technically simpler approach where one notices that at any singular point a small neck would need to form. Due to this the first Neumann eigenvalue would become very small, a fact that would contradict the lower bound provided by Theorem 3.3. This approach is described in [**1**]. As it will be useful later, we will take the former approach. We refer for also to [**?**] for another example of this method.

We first note the following fact which we state without proof

Lemma 3.13. *Fix $\epsilon > 0$ and $C_1 > 0$ and a three manifold M. Let Σ be a smooth oriented compact minimal surface in M. Moreover, suppose that the focal radius of Σ is at least ϵ and that Σ has uniformly bounded curvature:*

$$\sup_\Sigma |A^\Sigma| \leq C_1.$$

Suppose that $u \in C^\infty(\Sigma)$ is a function so that

$$\sup_\Sigma |u| + |\nabla u| + |\nabla^2 u| < \epsilon$$

and $\Gamma_u = \left\{ \exp^M(u(p)\mathbf{n}(p)) : p \in \Sigma \right\}$ the normal exponential graph of u is a smooth minimal surface. Then u satisfies an equation of the form

$$\Delta^\Sigma u + |A^\Sigma|^2 u + Ric(\mathbf{n}, \mathbf{n}) u + Q(u, \nabla u, \nabla^2 u) = 0$$

where Q satisfies

$$|Q(t, \mathbf{x}, B)| \leq C_2 \left(t^2 + |\mathbf{x}|^2 + |B|^2 \right)$$

and C_2 depends only on C_1 and ϵ.

Proof. (of 3.1) By Theorems 3.2 and 3.3 we know that Σ satisfies the conditions of Proposition 3.5. Thus, up to passing to a subsequence, there is a minimal embedded surface Σ so that Σ_i converge to Σ in a Hausdorff sense and there are a finite of points $\mathcal{S} \subset M$ so that $\Sigma_i \to \Sigma$ in $C^l_{loc}(M \backslash \mathcal{S})$ for any $l \geq 2$.

Since we assume $\pi_1(M) = 0$ we know that Σ along with every Σ_i is oriented. Fix any $\epsilon > 0$ chosen small enough so that for $x_i, x_j \in \mathcal{S}$ disjoint $B_{4\epsilon}(x_i) \cap B_{4\epsilon}(x_j) = \emptyset$ and so that the focal radius of Σ is larger then ϵ. For i sufficiently large (depending on ϵ) the convergence of Σ_i to Σ implies that there are smooth functions u_i^j defined on

$$\Sigma_\epsilon = \Sigma \backslash \cup_{x \in \mathcal{S}} \bar{B}_\epsilon(x)$$

so that

$$\Sigma_i \backslash \cup_{x \in \mathcal{S}} \bar{B}_\epsilon(x) = \cup_j \Gamma_{u_i^j}$$

and

$$\sup_{\Sigma_\epsilon} |u_i^j| + |\nabla^\Sigma u_i^j| + |(\nabla^\Sigma)^2 u_i^j| < \delta_i \leq \frac{1}{2}\epsilon$$

where here $\delta_i \to 0$. The embeddedness and orientablility imply that we can order the u_i^j from top to bottom

$$u_i^1 > u_i^2 > \ldots > u_i^N$$

and the ordering and number of sheets, N, is independent of ϵ. Define

$$w_i = u_i^1 - u_i^N > 0$$

to be the separation between the top and bottom sheet. Clearly, $w_i \in C^\infty(\Sigma_\epsilon)$ and satisfies

$$\sup_{\Sigma_\epsilon} |w_i| + |\nabla^\Sigma w_i| + |(\nabla^\Sigma)^2 w_i| < 2\delta_i \leq \epsilon.$$

Moreover, since one verifies that w_i satisfies the following equation

$$L^\Sigma w_i + Q(u_i^1, \nabla^\Sigma u_i^1, (\nabla^\Sigma)^2 u_i^1) - Q(u_i^N, \nabla^\Sigma u_i^N, (\nabla^\Sigma)^2 u_i^N) = 0$$

This gives the estimate

$$|L^{\Sigma}w_i| \leq C\delta_i \left(w_i + |\nabla^{\Sigma}w_i| + |(\nabla^{\Sigma})^2 w_i|\right)$$

for some constant C independent of i.

Fix a point $p \in \Sigma_{4\epsilon}$ and set

$$\hat{w}_i = \frac{w_i}{w_i(p)}$$

so that $\hat{w}_i(p) = 1$ and

$$|L^{\Sigma}\hat{w}_i| \leq C\delta_i \left(\hat{w}_i + |\nabla^{\Sigma}\hat{w}_i| + |(\nabla^{\Sigma})^2 \hat{w}_i|\right)$$

Since $\hat{w}_i > 0$ and satisfies the above differential inequality smallness of δ_i the L^2 theory and the Harnack inequality gives on $\Sigma_{\frac{3}{2}\epsilon}$ uniform C^{α} bounds for $\hat{w}_i$ and the L^2 norm of the right hand side tending uniformly to 0. Moreover, on this domain $\hat{w}_i$ is uniformly positive. Hence as $i \to \infty$ a subsequence converges uniformly in $\Sigma_{2\epsilon}$ to $\hat{w}$ which satisfies $\hat{w}(p) = 1$, $\hat{w} > 0$ and $L^{\Sigma}\hat{w} = 0$ weakly (and hence strongly).

We may also take ϵ to 0 and hence obtain a positive solution Jacobi function $\hat{w}$ on $\Sigma\backslash\mathcal{S}$. If we prove a removable singularities theorem for $\hat{w}$ we would be done. Rather than taking this approach we use the logarithmic cut-off trick once again. Note that by **??** that $\Sigma\backslash\mathcal{S}$ is a stable region. Indeed, for each $x_i \in \mathcal{S}$ let $r_i(x) = d^{\Sigma}(x, x_i)$. For $\epsilon > 0$ define ϕ^i_ϵ by

$$\phi^i_\epsilon = \begin{cases} 0 & r_i(x) < \epsilon \\ 2 - 2\frac{\log r_i(x)}{\log \epsilon} & \epsilon \leq r_i(x) \leq \sqrt{\epsilon} \\ 1 & r_i(x) > \sqrt{\epsilon} \end{cases}$$

We construct a function ϕ_ϵ with compact support in $\Sigma\backslash\mathcal{S}$ by taking the product. As Σ is a smooth minimal surface the monotonicity formula that for any $x \in \Sigma$ for $r > 0$ sufficiently small $Area(B_r(x) \cap \Sigma) \leq Cr^2$. Hence, we can estimate

$$\int_{\Sigma} |\nabla \phi_\epsilon|^2 \leq \frac{C}{|\log \epsilon|}$$

for some uniform C independent of ϵ. Notice the right hand side $\to 0$ as $\epsilon \to 0$. Plugging the hϕ_ϵ into the stability inequality for $\Sigma\backslash\mathcal{S}$ and letting $\epsilon \to 0$ gives a the desired contradiction. Hence the Σ_i converge to Σ with multiplicity one and we can appeal to Theorem 3.10 to get the smooth convergence to Σ everywhere. The curvature estimates are a simple consequence of this fact. $\square$

CHAPTER 4

Overview of Colding-Minicozzi Theory

1. Structure of embedded minimal disks

We are now going to begin the final part of the class – the discussion of the results of Colding and Minicozzi regarding the compactness properties of minimal surfaces with a uniform bound on the genus. From now on we will only discuss $\mathbb{R}^3$ though as before many (though not all!) of the results are local and so hold also in arbitrary three-manifolds. We note that in $\mathbb{R}^3$ there are no closed minimal surfaces. In particular, by the genus of a compact oriented surface Σ with boundary we mean the genus of the resulting compact surface obtained by gluing in disks along all boundary components of Σ. With this in mind we note that the work of Colding and Minicozzi does not require a bound on the Euler characteristic, only on the genus. In other, words surfaces minimal surfaces modelled an Riemann's examples (i.e. with infinite Euler characteristic and genus 0) can be understood within the framework of their theory.

The heart of the results of Colding and Minicozzi deals with understanding the structure of (compact) embedded minimal disks in $\mathbb{R}^3$. For this reason – and for the sake of technical simplicity– we will focus only on this case. To be precise, we are interested in the following question: Fix $R > 0$ and suppose that $0 \in \Sigma \subset B_{2R}(0) \subset \mathbb{R}^3$ is an embedded minimal disk, with $\partial\Sigma \subset \partial B_{2R}$. What is the structure of $\Sigma_{0,R}$? Based on classical examples of complete, embedded minimal surfaces we should expect at least two models: The plane – i.e. $\Sigma_{0,R}$ is relatively flat, for instance a graph over some plane; and the helicoid – i.e. $\Sigma_{0,R}$ should consist of an axial region of large curvature and a large number of flat sheets transverse to the axial region that join up into two sheets that spiral around the axis. The remarkable fact proved by Colding and Minicozzi is that these are the only possible models. Indeed, using this description of embedded minimal disks, Meeks and Rosenberg [**?**] showed that the only complete, properly embedded minimal disks in $\mathbb{R}^3$ were plane and helicoid.

Unfortunately, to make the above classification statement both precise and understandable is a rather hard. This is due to the difficulty in adequately describing in a quantitative way surfaces that "look like" the helicoid. For instance, unlike the helicoid there are (necessarily compact) embedded minimal disks for which the axis is curved. Even worse, unlike the helicoid, the axial region may have curvatures varying dramatically as one travels along it. This later possibility gives certain regions where the flat parts are very close together (i.e. correspond to a tightly spiralling helicoid) and other regions where the flat regions are relatively far apart (i.e. correspond to a slowly spiralling helicoid). As examples show [**?, ?**], this differing behavior need not be well separated.

Rather than dive into precise statements, lets begin with a outline of the proof: First of all we will study the structure of Σ near points of very large curvature. By Lemma 2.28, near any such point there will be a C blow-up pair (p, s). Moreover, if the curvature is very large then C will itself be very large. As we will see it is necessary to take C very large. Previously, we have used a priori control on the *geometry* of the surface to rule out the existence of such blow-up pairs. However, in our situation we only have a priori control on the *topology* of the surface (namely it is an embedded disk) and so cannot rule out the existence of such a pair (indeed based on the helicoid expect them). However, by working always on the scale of s, we will be able to show that near p (i.e. within $B_{C_1 s}(p)$) Σ will contain at least one multi-valued near p and of a size proportional to s. This is precisely what one obtains for a blow-up pair lying on the axis of the helicoid. This allows us to split the surface into two regions a "axial" region $\mathcal{A}$ roughly corresponding to the union of the balls $B_{C_1 s_i}(p_i)$ where (p_i, s_i) are all C blow-up pairs and a "flat" region which corresponeds to the rest of the surface $\mathcal{F} = \Sigma_{0,R} \backslash \mathcal{A}$. Within $\mathcal{A}$ we have some local structure for Σ (namely the existence of the multi-valued graph) while in the other region one has curvature bounds and so also has local structure (namely a uniform scale on which the surface is a graph). The global structure of these regions is at this stage is undetermined. In a scale one helicoid the region $\mathcal{A}$ would consists of a fixed size tubular neighborhood of the axis and $\mathcal{F}$ would be the two sheets spiralling around this region.

The second step is to show that for any C blow-up pair (p, s) in Σ the multi-valued graph that one finds near p can be extended, as a multi-valued graph, almost all the way to boundary. The idea here is to use the surface Σ as a barrier with which to find *stable* minimal disks that that spiral in between the multi-valued graphs initially and that connect out to the boundary. After some careful analysis, one verifies that the initial spiralling of the stable surface propagates outward and the stable surface "traps" the original multi-valued graph and in doing so forces it to remain a multi-valued graph. This prevents the axial set $\mathcal{A}$ from filling up too large a region in Σ. Since Σ divides $B_R(0)$, the above argument can be adapted to verify that there must be a second multi-valued graph spiralling between the one found. In fact it can be shown that there are no other sheets.

The third step is to show that there are "enough" blow-up points. In other words if the singular set is non-empty it must be "large" – from now on we assume the singular set is non-empty as otherwise we are in the "plane" case. This will follow by noting that for a given blow-up pair (p, s). the two multi-valued graphs that start near p and go nearly to the boundary from step 2 give a weak notion of "above" and "below" relative to p. Using this one is able to show that there must be a blow-up pair above (p, s) and one below and near to p (as measured relative to the scale s). This fact can be iterated to find a large "backbone" of blow-up pairs. By considering the associated multi-valued graphs one obtains a "skeleton" for Σ. This shows that qualitatively $\mathcal{A}$ looks like the tubular neighborhood of a curve, though at this stage we lack quantitative control on $\mathcal{S}$.

The fourth step is to use the skeletal structure of discovered above to prove the one-sided curvature estimate. That is to show that an embedded minimal disk that comes close to, but remains on one side of, a fixed plane must have a priori interior curvature bounds. The idea is that the skeletal structure implies that if the curvature were large (and hence there were at least one point in $\mathcal{A}$) the disk would have to "fill space" and so couldn't remain with in the half-space as claimed.

The final step is to use to one-sided curvature estimate to show that the "backbone" of singular points is reasonably straight (i.e. has a uniform cone property) and to show that the rest of the disk consists of two multi-graphs formed by extending the multi-graphs coming out of the "backbone" into precisely two sheets that spiral through each other and around the "backbone". This is the structure one expects for a "helicod"-like surface.

2. Applications of Colding and Minicozzi's results

In [**?**] Colding and Minicozzi apply their structural results to prove the following compactness result for embedded minimal disks in $\mathbb{R}^3$ which shows that for sequences of embedded minimal disks with boundaries tending to ∞ then either the curvature is uniformly bounded along a subsequence or the convergence is singular in a manner modeled on the homothetic blow-down of the catenoid:

THEOREM 4.1. *Let $\Sigma_i \subset B_{R_i} = B_{R_i}(0) \subset \mathbb{R}^3$ be a sequence of embedded minimal disks with $\partial\Sigma_i \subset \partial B_{R_i}$ where $R_i \to \infty$. If $\sup_{B_1\cap\Sigma_i}|A|^2 \to \infty$ then there exists a sub-sequence, Σ_j, and a Lipschitz curve $\mathcal{S} : \mathbb{R} \to \mathbb{R}^3$ such that after a rotation of $\mathbb{R}^3$:*

(1) $x_3(\mathcal{S}(t)) = t$.
(2) *Each Σ_j consists of exactly two multi-valued graphs away from $\mathcal{S}$ (which spiral together).*
(3) *for each $1 > \alpha > 0$ $\Sigma_j\backslash\mathcal{S}$ converges in the C^α-topology to the foliation, $\mathcal{F} = \{x_3 = t\}_t$ of $\mathbb{R}^3$.*
(4) $\sup_{B_r(\mathcal{S}(t))\cap\Sigma_j}|A|^2 \to \infty$ *for all $r > 0, t \in \mathbb{R}$.*

Notice that away from the singular set $\mathcal{S}$, the convergence is smooth on compact subsets, whereas at the singular set the curvature must blow-up. Also note that the assumption that $R_i \to \infty$ is essential, as is shown by examples constructed by Colding and Minicozzi in [**?**] (see also Chapter **??** and in particular Figure **??**). One would hope to deduce Theorem 4.1 directly from the description of embedded disks given in Section **??** with $\mathcal{A}_i \to \mathcal{S}$ in Hausdorff distance. However, it is not this easy because the description of embedded disks is a local statement, whereas the compactness theorem is global in nature – a point made clear by the examples of [**?**]. The results needed to bridge this gap can be found in [**?**]. Roughly speaking, the problem is that the scales of blow-up pairs can a priori change dramatically and indeed if the R_i stay bounded the axial while containing the singular set need not collapse to it.

In another direction, their work allows for one to classify complete properly embedded minimal disks in $\mathbb{R}^3$:

THEOREM 4.2. *(Meeks and Rosenberg* [**5**]*) Let Σ be a complete properly embedded minimal disk in $\mathbb{R}^3$ then Σ is either a plane or a helicoid.*

REMARK 4.3. This theorem implies that the singular set in the limit above must be a straight line orthogonal to the foliation.

A final example is the resolution of the following so called Calabi-Yau question:

THEOREM 4.4. *(Colding and Minicozzi* [**4**]*) Let Σ be a complete embedded minimal surface of finite topology in $\mathbb{R}^3$, then Σ is properly embedded.*

Remark 4.5. If the surface is allowed to be immersed then this theorem fails as examples of Jorge-Xavier and Nadirashvilli show.

CHAPTER 5

Embedded Minimal Disks Near Blow-up pairs

We begin by analyzing the structure of embedded minimal disks near points of large curvature via blow-up arguments.

The goal is to prove the following theorem (Theorem 0.4 of [**2**]):

THEOREM 5.1. *Given $N, \omega > 1$ and $\epsilon > 0$ there exists a $C = C(N, \omega, \epsilon) > 0$ so that:*

For $0 \in \Sigma$ an embedded minimal disk in $\mathbb{R}^3$ with $\partial\Sigma \subset \partial B_R$ if for some $0 < r_0 < R$ one has that $(0, r_0)$ is a C blow-up pair in Σ then there is a $\bar{R} < \frac{r_0}{\omega}$ and (after a rotation of $\mathbb{R}^3$) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega\bar{R}} \backslash D_{\bar{R}}$ with gradient less then ϵ and $\mathrm{dist}^{\Sigma}(0, \Sigma_g) \leq 4\bar{R}$.

What we mean by Σ_g being an N-valued graph (multi-graph for short) over $D_{r_1} \backslash D_{r_0}$ is that there is a function u on $S^{-N\pi, N\pi}_{r_1, r_0} := [r_1, r_0]_r \times [-N\pi, N\pi]_\theta$ so that

$$\Sigma_g = \Gamma_u := \left\{ (r\cos\theta, r\sin\theta, u(r,\theta)) : (r,\theta) \in S^{-N\pi, N\pi}_{r_1, r_0} \right\}.$$

That is we consider the graph of a function over polar coordinate domain.

The outline of the proof is as follows:

(1) First relate the total curvature and area of intrinsic balls in minimal disks using the Gauss-Bonnet formula.
(2) Show that a bound on the total curvature of a minimal disk gives a pointwise curvature bound. These fact together with the former will imply that intrinsic balls centered at a blow-up point have faster than quadratic area growth.
(3) Prove that under a uniform curvature bound if disjoint minimal surfaces come close to each other then they must be almost stable. In particular, they are quite flat.
(4) Use this to give an explicit a priori area bound in a fixed extrinsic ball for an embedded minimal disk that has curvature uniformly bounded by 4. Importantly, this bound will be better then the straight forward one arising from volume comparison arguments.
(5) Finally, the faster then quadratic area growth for balls centerd at a blow-up point will imply that there are many flat regions of the disk at least one of which will be our desired multi-graph.

1. Area and total curvature in embedded disks

We begin with a simple consequence of the Gauss-Bonnet formula:

LEMMA 5.2. *Let Σ be an embedded minimal disk with $\mathcal{B}^{\Sigma}_{r_0}(p) \subset \Sigma \backslash \partial\Sigma$, then*

$$Length(\partial \mathcal{B}^{\Sigma}_{r_0}(p)) - 2\pi r_0 = \frac{1}{2}\int_0^{r_0}\int_{\mathcal{B}_r}|A^{\Sigma}|^2$$
$$Area(\mathcal{B}^{\Sigma}_{r_0}) - \pi r_0^2 = \frac{1}{2}\int_0^{r_0}\int_0^{r}\int_{\mathcal{B}_\rho}|A^{\Sigma}|^2.$$

PROOF. By the first variation formula and Gauss-Bonnet formula and the fact that Σ is a disk one has:

$$\frac{d}{dt}\int_{\partial \mathcal{B}^{\Sigma}_{r_0}(p)} 1 = \int_{\partial \mathcal{B}^{\Sigma}_{r_0}(p)} k_g = 2\pi - \int_{\mathcal{B}^{\Sigma}_{r_0}(p)} K^{\Sigma}.$$

The proof follows then from the Gauss equations and the co-area formula. □

COROLLARY 5.3. *For Σ as above one has*

$$Area(\mathcal{B}^{\Sigma}_{r_0}(p)) \leq \pi r_0^2 + \frac{1}{4} r_0^2 \int_{\mathcal{B}^{\Sigma}_{r_0}(p)} |A|^2 \tag{1.1}$$

and

$$t^2\int_{\mathcal{B}^{\Sigma}_{r_0-2t}(p)}|A|^2 \leq \frac{1}{2}\int_{\mathcal{B}_{r_0}}|A|^2(r_0-r)^2 \tag{1.2}$$

$$= 2(Area(\mathcal{B}^{\Sigma}_{r_0}(p)) - \pi r_0^2) \leq r_0 Length(\partial\mathcal{B}^{\Sigma}_{r_0}(p)) - 2\pi r_0^2 \tag{1.3}$$

PROOF. The first estimate is immediate. The second follows from the co-area formula and integration by parts we refer to Corollary 1.7 of [**2**] for the details. □

We now prove a result due to Schoen and Simon that a embedded minimal disk with bounded total curvature admits a point-wise curvature estimate. First a technical lemma

LEMMA 5.4. *given $C > 0$ there exists an $\epsilon > 0$ so that if $\mathcal{B}_{9s}(0) \subset \Sigma \subset \mathbb{R}^3$ is an embedded minimal disk with*

$$\int_{\mathcal{B}_{9s}}|A|^2 \leq C \text{ and } \int_{\mathcal{B}_{9s}\backslash\mathcal{B}_s}|A|^2 \leq \epsilon$$

then

$$\sup_{\mathcal{B}_s}|A|^2 \leq s^{-2}$$

PROOF. Recall that for $\epsilon > 0$ sufficiently small Theorem 2.26 gives that

$$\sup_{\mathcal{B}_{8s}\backslash\mathcal{B}_{2s}}|A|^2 \leq C_1^2 s^{-2}$$

for some uniform C_1. On the other hand by (1.1) we have a uniform bound on the area of $\mathcal{B}_{9s}(0)$ depending only on the given C. In particular, we can argue as in Lemma 3.7 that $\partial\mathcal{B}_{2s}$ has a bound on its length given by $C_2 s$ for some C_2 depending only on C. In particular, the arguments of that lemma imply that for ϵ sufficiently small, after an ambient rotation $\partial\mathcal{B}_{2s}$ can be written as a graph of a function u with C^2 norm bounded by $C_2\epsilon^{1/2}$ and over a (a priori immersed) curve σ lying in the plane $\{x_3 = 0\}$. Notice this is slightly different from Lemma 3.7 as in that case we were treating extrinsic balls and so had extrinsic control.

We claim that (after a translation) for ϵ sufficiently small that $\sigma \subset \mathbb{R}^2 \backslash D_{2s}$. To see this we pick a point $y \in \partial \mathcal{B}_2$ and consider γ_y the geodesic segment parameterized by arclength connecting y to $\partial \mathcal{B}_{8s}$ that lies in the geodesic connecting 0 and y. By integrating the pointwise curvature estimate we see that along γ_y one has

$$\sup_{t,t' \in [0,6s]} |\gamma'(t) - \gamma'(t')| \leq C_3 \epsilon^{1/2}$$

(here we are taking the norm as vectors in $\mathbb{R}^3$). In particular, γ_y is almost a straight euclidean line.

Hence, by ensuring that $\mathbf{n} = \mathbf{e}_3$ at some point on $\partial \mathcal{B}_{2s}$ we see that (after translating) for ϵ sufficiently small $\partial \mathcal{B}_{8s}$ lies outside of the cylinder $D_{3s} \times \mathbb{R}$. Notice that $\mathcal{B}_{2s} \subset B_{2s}(0)$ hence for ϵ small if we consider $\partial D_{5/2s} \times \mathbb{R} \cap \mathcal{B}_{8s}$ then we get a collection of immersed curves each of which is a multi-valued graph over ∂D_{2s}. Since, Σ is embedded these curves $\gamma_1, \dots, \gamma_n$ are actually embedded and hence to close up must be actual be graphs. Since each γ_i bounds a topological disk in $\mathcal{B}_{8s}$ by the maximum principle they bound a disk in $D_{2s} \times \mathbb{R}$. By Rado's theorem each of these components must be a graph (since the γ_i are graphs over circles). Since graphs are stable and one of these components contains 0 we can appeal to Schoen's curvature estimate to get a priori bound on the curvature. The desired bound follows by possibly shrinking ϵ (either directly or via a compactness argument).

□

We use this to prove an intrinsic version of a result due to Schoen and Simon

Corollary 5.5. *Given a constant C_I there is a C_P so that if $\mathcal{B}_{2s}(0) \subset \Sigma \subset \mathbb{R}^3$ is an embedded minimal disk with*

$$\int_{\mathcal{B}_{2s}} |A|^2 \leq C_I$$

then

$$\sup_{\mathcal{B}_s} |A|^2 \leq C_P s^{-2}.$$

Proof. Given $C = C_I$ let ϵ be the constant of the preceding theorem and set $N > C_I/\epsilon$ an integer. Given $x \in \mathcal{B}_s$ by a pigeonhole argument there exists a $1 \leq j \leq N$ with

$$\int_{9^{1-j}s(x)} \backslash \mathcal{B}_{9^{-j}s(x)} |A|^2 \leq C_I/N \leq \epsilon.$$

As a consequence by the preceding theorem

$$|A|^2(x) \leq (9^{-j}s)^{-2} \leq 9^{2N} s^{-2}.$$

□

This theorem will be used to give large total curvature around a C-blow up pair when C is large.

2. Almost Stability

As we saw in the proof of Theorem 3.1 when we had two disjoint embedded minimal surfaces collapsing to a single sheet the resulting limit object was stable. In fact a similar result holds without going to the limit whenever one has a priori curvature bounds.

Let $\Sigma \subset \mathbb{R}^3$ be an oriented minimal surface. For $\alpha \geq 0$ we introduce the α-stability operator

$$L_\alpha^\Sigma = \Delta^\Sigma + (1-\alpha)|A|^2.$$

Notice that for $\delta = 0$ this is the usual stability operator. We say a domain $\Omega \subset \Sigma$ is α-stable if for all $\phi \in C_0^\infty(\Omega)$ one has

$$\int_\Omega \phi L_\alpha^\Sigma \phi \leq 0.$$

In particular, Ω is α-stable if and only if we have a α-stability inequality

$$(1-\alpha)\int_\Sigma |A|^2\phi^2 \leq \int_\Sigma |\nabla\phi|^2.$$

REMARK 5.6. We remark that for α sufficiently small ($\alpha \leq \frac{1}{4}$) that our proof of 2.21 applies to α-stable minimal surfaces.

Recall the stability operator was the linearization of the minimal surface equation. The advantage of L_δ is that it allows us to absorb the quadratic term for small perturbations when there is a uniform curvature bound:

PROPOSITION 5.7. *There exists a $\delta > 0$ so that if Σ is minimal and $u > 0$ satisfies*

$$|\nabla u| + |u||A| \leq \delta$$

on $\Omega \subset \Sigma$ and

$$\Sigma_u = \Gamma_u := \{\mathbf{x}(p) + u(p)\mathbf{n}(p) : p \in \Sigma\}$$

is a minimal surface then Ω is $\frac{1}{2}$-stable.

PROOF. The proof follows from two facts. First of all, for $\delta \leq \frac{1}{2}$ we have that Γ_u is within the focal radius. The mean value theorem from calculus applied to the minimal surface equation implies that if $w = \log u$ then w satisfies

$$\Delta^\Sigma w = -|\nabla^\Sigma w|^2 + \text{div}\,(a\nabla^\Sigma w) + \langle \nabla^\Sigma w, a\nabla^\Sigma w\rangle + \langle \mathbf{b}, \nabla^\Sigma w\rangle + (c-1)|A|^2$$

where $a = a_{ij}, \mathbf{b} = b_j$ and c are smooth functions on Σ satisfying $|a|, |c| \leq 3|A||u| + |\nabla u|, |\mathbf{b}| \leq 2|A||\nabla u|$.

For any cut-off $\phi \in C_0^\infty(\Omega)$ one argues as in Lemma **??**. Namely by multiplying the above equation by ϕ and integrating by parts and using the estimates $|a|, |c| \leq 3\delta, |\mathbf{b}| \leq 2\delta|\nabla w|$ one eventually arrives at

$$(1-3\delta)\int_\Sigma \phi^2|A|^2 \leq (5\delta - 1)\int_\Sigma \phi^2|\nabla w|^2 + 2(1+3\delta)\int_\Sigma |\phi\nabla w||\nabla\phi|.$$

The proposition then follows from the absorbing inequality and taking δ sufficiently small. □

Recall that given an a priori curvature bound, there is a uniform scale on which any surface may be written as a graph. To be precise: suppose that $1 \geq \rho_G > 0$ is the uniform scale so that if Σ is a surface with with $|A| \leq 4$ on $\mathcal{B}_2(p) \subset \Sigma \backslash \partial\Sigma$ then $\mathcal{B}_{\rho_2}$ can be written as a graph u over $T_p\Sigma$ with

$$|\nabla u| + \rho_G |\nabla^2 u| \leq \frac{1}{10}.$$

It is straight forward to see that if there are two disjoint surfaces Σ_1 and Σ_2 each with an a priori curvature bound and so that $\mathcal{B}^{\Sigma_i}_{r_0}(p_i) \subset \Sigma_i \backslash \partial\Sigma_i$ and p_1 and p_2 are close extrinsically then these two graphs must be almost parallel. That is

LEMMA 5.8. *There exists a $C_0, \rho_0 > 0$ so that the following holds: If $\rho_1 \leq \min\{\rho_0, \rho_G\}$ and Σ_1, Σ_2 are oriented minimal surfaces with $|A|^2 \leq 4$ and so that there are $p_i \in \Sigma_i$ with*

(1) $\mathcal{B}^{\Sigma_i}_{4\rho_G} \subset \Sigma_i \backslash \partial\Sigma_i$
(2) $p_2 \in B_{\rho_1}(p_1) \cap \Sigma_2$
(3) $\mathcal{B}_{2\rho_G}(p_2) \cap \mathcal{B}_{2\rho_G}(p_1) = \emptyset$

Then $\mathcal{B}_{\rho_G}(x)$ is the normal exponential graph of a function u over a domain containing $\mathcal{B}_{\rho_G/2}(x)$ with $u \neq 0$ and

$$|\nabla u| + 4|u| \leq C_0 \rho_1.$$

REMARK 5.9. We emphasize that we have broken scaling invariance by our curvature assumption and our choice of ρ_G. That is why the conclusions of the theorem are *not* scaling invariant.

PROOF. The proof follows from Lemma 2.1 and the observation above about disjoint surfaces and a straight forward compactness argument. In particular, under the curvature assumption we may take $\rho_1 \to 0$ and obtain two embedded minimal surfaces Σ_1 and Σ_2 so that Σ_1 lies on one side of Σ_2 and the two surfaces meet at at a point. By the strong maximum principle the two surfaces must coincide and hence the theorem follows. □

As consequence, under an a priori curvature bound disjoint minimal surfaces that are near at some interior point must be δ-stable–we also make it a scale invariant statement:

COROLLARY 5.10. *Given $C_0, \delta > 0$ there exists an $\epsilon(C_0, \delta) > 0$ so that if $p_i \in \Sigma_i$ are embedded minimal disks with*

(1) $\Sigma_1 \cap \Sigma_2 = \emptyset$,
(2) $\mathcal{B}_{2R}(p_i) \cap \partial\Sigma_i = \emptyset, |p_1 - p_2| \leq \epsilon R$,
(3) $\sup_{\mathcal{B}_{2R}(p_i)} |A|^2 \leq C_0 R^{-2}$,

then $\mathcal{B}_R(p_i)$ is δ-stable.

REMARK 5.11. This holds more generally provided one is willing to go to the universal cover.

PROOF. We rescale by $\frac{R}{\sqrt{C_0}}$ so that the curvature of the Σ_i are uniformly bounded by 4. After doing so our surfaces can be taken to lie in a fixed size ball and so a compactness argument as above can be used. □

REMARK 5.12. Both of the above results also follow from the Harnack inequality–though this is a bit misleading as we do not need the full power of such an estimate.

3. A priori estimates near blow-up points

We now show a priori area and total curvature estimates around C blow-up pairs (p,s) (on the scale of the blow-up pair) – it will be important that these bounds are independent of C and that are better than what can be obtained by comparison arguments. The idea is to rescale by Cs^{-1} so one obtains uniform curvature bounds on large balls (at least when C is large).

LEMMA 5.13. *There is a $C_1 > 0$ so that: If $0 \in \Sigma \subset B_{2R}$ is an embedded minimal surface with $\partial\Sigma \subset \partial B_{2R}$ and $|A|^2 \leq 4$ then there exist $1/2$-stable domains $\Omega_j \subset \Sigma$ and a function $0 \leq \chi \leq 1$ on Σ vanishing on $B_R \cap \Sigma \setminus \cup_j \Omega_j$ so that*

$$Area(\{\chi < 1\} \cap B_R) \leq C_1 R^3$$

and

$$\int_{\mathcal{B}_R} |\nabla \chi|^2 \leq C_1 R^3.$$

PROOF. We may assume that $R > \frac{1}{2}\rho_G$ as otherwise $B_R \cap \Sigma$ is stable. Now pick $\delta > 0$ from Proposition 5.7 and choose C_0, ρ_0 from Lemma 5.8. Set

$$\rho_1 = \min\{\rho_0/C_0, \delta/C_0, \rho_G/4\}$$

For $x \in B_{2R-\rho_1} \cap \Sigma$ let B_x^+ be the component of $B_{\rho_1}(x) \setminus \Sigma_{x,\rho_1}$ into which $\mathbf{n}(x)$ points. Set

$$VB = \left\{x \in B_R \cap \Sigma : B_x^+ \cap \Sigma \setminus \mathcal{B}_{4\rho_1}(x) = \emptyset\right\}$$

and let $\{\Omega_j\}$ be the components of $B_R \cap \Sigma \setminus \overline{VB}$. Choose a maximal disjoint collection $\{\mathcal{B}_{\rho_1}(y_i)\}_{1\leq i \leq N}$ of balls centered in VB. Thus, $\{\mathcal{B}_{2\rho_1}(y_i\}_{1\leq i\leq N}$ is a covering of VB. Moreover, the half-balls $B_{\rho_1/2}(y_i) \cap B_{y_i}^+$ are pair-wise disjoint. Indeed, if not then there would be (up to a relabelling) distinct $y_1, y_2 \in VB$ with $|y_1 - y_2| < \rho_1$. By the definition of of VB and the triangle inequality we must have $y_2 \in \mathcal{B}_{4\rho_1}(y_1)$. As Σ is flat on scale ρ_1, we have also that $y_2 \in \Sigma_{y_1,\rho_1}$ in other words $y_2 \in \mathcal{B}_{\rho_1}(y_1)$ a contradiction.

By the flatness of Σ on the scale of ρ_1 we have that $B_{\rho_1/2}(y_i) \cap B_{y_i}^+$ has volume a uniform multiple of ρ_1^3 for each i. Since each such "half-ball" is contained in B_{2R} a pigeonhole argument gives that $N \leq CR^3$ for some constant C.

Let us now define the function χ by

$$\chi(x) = \left\{ \begin{array}{cc} 0 & x \in VB \\ d^\Sigma(x, VB)/\rho_1 & x \in \mathcal{T}_{\rho_1}(VB) \setminus VB \\ 1 & d^\Sigma(x, VB) \geq \rho_1. \end{array} \right.$$

Since $\mathcal{T}_{\rho_1}(VB) \subset \cup_{i=1}^N \mathcal{B}_{3\rho_1}(y_i)$, $|A|^2 \leq 4$ and $N \leq CR^3$ the area estimate for graphical regions give the desired area bound. Moreover, since $|\nabla\chi| \leq \rho_1^{-1}$ this area bound also gives the Dirichlet energy bound.

It remains to verify that the Ω_j are each $\frac{1}{2}$-stable. To that end we set

$$T_{\rho_1}^+(\Omega_j) = \left\{y \in \mathbb{R}^3 : y = x + t\mathbf{n}(x), x \in \Omega_j, t \in [0, \rho_1)\right\}$$

be the one sided tubular neighborhood of Ω_j. By definition for any fixed j, the $T_{\rho_1}^+(\Omega_j) \cap \Sigma \setminus \Omega_j$ must have Hausdorff distance ρ_1 to Σ. That is this set doesn't have any big gaps. We claim in fact that one of the components of this set must in fact be a the normal exponential graph of a function $u_j > 0$ on Ω_j satisfying $|\nabla u_j| + 4|u_j| \leq \delta$. Hence, Proposition 5.7 implies that Ω_j is $\frac{1}{2}$-stable.

We now wish to justify our claim. To that end we note that the local graph Lemma 5.8 implies that for each point y of $T^+_{\rho_1}(\Omega_j)\cap\Sigma\backslash\Omega_j$, $\mathcal{B}_{\rho_G}(y)$ can be written as a graph u_y over a domain in Ω_j that satisfies the bounds $|\nabla u_x|+4|u_x| \leq \min\{\delta, \rho_0\}$. The claim is verified if we can find a component where these graphs can be patched together. Clearly, there is a partial ordering by height on the components of $T^+_{\rho_1}(\Omega_j)\cap\Sigma\backslash\Omega_j$. One verifies by using Lemma 5.8 and the embeddedness of Σ that there is a unique component closest to Ω_j and that this component is as claimed. □

REMARK 5.14. We point out that the regions Ω_j are shown to be $\frac{1}{2}$-stable by showing (from the construction) that they are close to another piece of Σ at all of their points.

We can now prove the claimed a priori bounds. The idea is that away from a "negligible" set the that roughly corresponds to the axis of the helicod (i.e. where $\chi < 1$ in the above Lemma) the surface consists of many $\frac{1}{2}$-stable regions. For the purposes of total curvature estimates, $\frac{1}{2}$-stable is as good as stable and so these regions are rather flat and in particular have area growth comparable to a disk. As such we need only control the number of such regions and claim that the Dirichlet energy of χ does this.

LEMMA 5.15. *There exists a $C_1 > 0$ so that if $0 \in \Sigma \subset B_{2R}$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{2R}$ and $|A|^2 \leq 4$ then*

$$\frac{1}{2}\int_{\mathcal{B}_R} |A|^2(R-r)^2 = 2(Area(\mathcal{B}_R - \pi R^2) \leq 6\pi R^2 + 20C_1R^5.$$

REMARK 5.16. The a priori curvature bound gives, by the Bishop-Gromov volume comparison, that the area grows at most exponentially (i.e. like $e^{\Lambda R}$) the polynomial growth is a significant improvement and will give "doubling" estimates for the area and total curvature on large scales. By considering the helicoid the area growth can't be less than R^3.

PROOF. We choose C_1, χ and $\{\Omega_j\}$ as in the preceding Lemma. Define ψ on $\mathcal{B}_R$ to be the cut off

$$\psi(p) = 1 - r(p)/R = 1 - d^{\Sigma}(0,p)/R$$

so $\chi\psi$ vanishes away from $\cup_j\Omega_j$. By using $\chi\psi$ in the $\frac{1}{2}$-stability inequality one estimates

$$\begin{aligned}
\int_\Sigma |A|^2\chi^2\psi^2 &\leq 2\int_\Sigma |\nabla(\chi\psi)|^2 \\
&= 2\int_\Sigma \chi^2|\nabla\psi|^2 + 2\chi\psi\nabla\chi\cdot\nabla\psi + \psi^2|\nabla\chi|^2 \\
&\leq 6C_1R^3 + 3\int_\Sigma \chi^2|\nabla\psi|^2 \\
&\leq 6C_1R^3 + 3R^{-2}Area(\mathcal{B}_R)
\end{aligned}$$

where we used the absorbing inequality, the bound for the Dirichlet energy of χ and the fact that $|\nabla\psi|^2 \leq R^{-2}$.

Since $|A| \leq 4$ the area estimate for $\chi < 1$ and the above gives

$$\int_\Sigma |A|^2\psi^2 \leq 4C_1R^3 + \int_\Sigma |A|^2\chi^2\psi^2 \leq 10C_1R^3 + 3R^{-2}Area(\mathcal{B}_R).$$

However, if we go back to the proof of the Cacciopli identity we see that

$$\int_\Sigma |A|^2\psi^2 = 4(Area(\mathcal{B}_R - \pi R^2)$$

Since $3 < 4$ the area estimate follows. □

4. Concluding the proof

We next use the polynomial volume growth to show that one has on large balls a fixed doubling. This will be important in showing a quantitative flatness needed to find the multi-valued graph. One important point to take away is that in contrast to the uniform area bounds of Lemma 5.15 the doubling estimate will be scale invariant.

LEMMA 5.17. *There is a C_2 so that for $\beta, R_0 > 1$ there is a R_2 so that: If $0 \in \Sigma \subset \mathcal{B}_{R_2}$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{R_2}$ and*

$$\sup_\Sigma |A|^2 \leq 4|A|^2(0) = 4$$

then there is a $R_0 < R < \frac{R_2}{4\beta}$ with

$$\int_{\mathcal{B}_{3R}} |A|^2 + \beta^{-10}\int_{\mathcal{B}_{2\beta R}} |A|^2 \leq C_2 R^{-2} Area(\mathcal{B}_R)$$

REMARK 5.18. The lemma is stated in this form for technical purposes, we note that the estimate is essentially equivalent to

$$(2R)^{-2}Area(\mathcal{B}_{2R}) \leq C_2 R^{-2} Area(\mathcal{B}_R)$$

which makes the doubling clearer

PROOF. Set $\mathcal{A}(s) = Area(\mathcal{B}_s)$. Given an $m > 0$, the polynomial volume growth and lower bound on area growth given by the "Poincare" inequality gives

$$\left(\min_{1\leq n\leq m} \frac{\mathcal{A}((4\beta)^{2n}R_0)}{\mathcal{A}((4\beta)^{2n-2}R_0)}\right)^m \leq \frac{\mathcal{A}((4\beta)^{2m}R_0)}{\mathcal{A}(R_0)} \leq C_1'(4\beta)^{10m}R_0^3.$$

Now fix m so that $C_1' R_0^3 < 2^m$ and set $R_2 = 2(4\beta)^{2m}R_0$. The above estimate gives an n so that if $R_1 = (4\beta)^{2n-2}R_0$ and so that

$$\frac{\mathcal{A}((4\beta)^2 R_1)}{\mathcal{A}(R_1)} \leq 2(4\beta)^1 0.$$

i.e. we obtain a disk with fixed doubling. Without loss of generality we may take $\beta = 4^q$ for q a positive integer. We then obtain a j with $0 \leq j \leq q$ so that one has a fixed doubling for the area:

$$\frac{\mathcal{A}(4^{j+1}R_1)}{\mathcal{A}(4^j R_1)} \leq \left(\frac{\mathcal{A}(4\beta R_1)}{\mathcal{A}(R_1)}\right)^{1/(q+1)} \leq 2^{1/(q+1)}4^{10}.$$

If we set $R = 4^j R_1$ then this fixed doubling together with our Cacciopoli inequality and the previous estimate proves the lemma. □

Given a closed subset $\gamma \subset \partial B_r(p) \subset \Sigma$ we define the intrinsic sector over γ by

$$S_R(\gamma) := \left\{\exp_p(v) | r \leq |v| \leq r + R, \exp_p(rv/|v|) \in \gamma\right\}.$$

On helicoid, when p lies on the axis, intrinsic sectors can be chosen to roughly correspond to n-valued graphs. The goal will be to prove a comparable result for some of the intrinsic sectors of an arbitrary embedded minimal disk (as usual only near a blow-up pair and on the right scale). The doubling estimate will allows us to show that as one goes further out one can find more and more sectors with a scale invariant uniform curvature bound. By scaling things down to scale one, these sectors have to come close to each other and hence (some subset of them) will be nearly stable.

LEMMA 5.19. *Given $m > 0$, $\Omega > 1$: there are $C \geq 2\Omega R > 0$ and $C_1 > 0$ so: If $0 \in \Sigma$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_C$, $|A|^2(0) = 1$ and $|A|^2 \leq 1$ on $B_C \cap \Sigma$ then there is a connected arc $\gamma \subset \partial\mathcal{B}_R(0)$ so*

(1) *Length$(\gamma) = mR$*

(2) *$\int_\gamma k_g \leq C_1 m$.*

and so that $S_{\Omega R}(\gamma)$ is $\frac{1}{2}$-stable

REMARK 5.20. The conditions on γ roughly correspond to it being a multiplicity $M = m/2\pi$ copy of the disk of radius R. Thus the lemma states that we can find a large M-sheeted $\frac{1}{2}$-stable region in Σ.

PROOF. By Corollary 5.5 and (1.2) the length of ∂B_r grows faster than linearly. In other words, for any a_0 large there is an $R_0 = R_0(a_0)$ so that for $R_3 \geq R_0$ one has:

$$\frac{1}{2}a_0 R_3 \leq \frac{R_3}{4}\int_{\mathcal{B}_{R_3/2}} |A|^2 \leq Length(\partial B_{R_3})s.$$

Set $\beta = 2\Omega > 2$. By (1.2) and Lemma 5.17 there is an $R_2 = R_2(R_0, \beta)$ so that if $C \geq R_2$ then there is $R_0 < R_3 < \frac{R_2}{4\beta}$ so that

$$\int_{\mathcal{B}_{3R_3}} |A|^2 + \beta^{-10}\int_{\mathcal{B}_{2\beta R_3}} |A|^2 \leq C_2 R_3^{-2} Area(\mathcal{B}_{R_3})$$
$$\leq C_2(2R_3)^{-1} Length(\partial B_3).$$

Pick an n so that

$$a_0 R_3 \leq 4mnR_3 < Length(\partial B_{R_3}) \leq 8mnR_3$$

which we can do by our growth assumption on the length. Fix $2n$ disjoint arcs $\tilde{\gamma}_i \subset \partial B_{R_3}$ each with length $2mR_3$ with these arcs let

$$\tilde{S}_i = \{\exp_0(v) | 0 < |v| \leq 2\beta R_3, \exp_0(R_3 v/|v|) \in \tilde{\gamma}_i\}$$

i.e. sectors that go up to 0. The $\tilde{S}_i$ are disjoint by construction and so the doubling estimate and bound on $Length(\partial B_{R_3})$ give

$$\sum_{i=1}^{2n}\left(\int_{\mathcal{B}_{3R_3}\cap\tilde{S}_i} |A|^2 + \beta^{-10}\int_{\tilde{S}_i} |A|^2\right) \leq 4C_2 mn.$$

By the pigeonhole principle and a relabelling we conclude that

$$\left(\int_{\mathcal{B}_{3R_3}\cap\tilde{S}_i} |A|^2 + \beta^{-10}\int_{\tilde{S}_i} |A|^2\right) \leq 4C_2 m.$$

(i.e. we get a bound independent of n on at least half of the sectors). Notice this is a *uniform* bound. In particular, we are free to take n as large as we like provided that C is large enough and still get the same bound. This is because increasing n is the same as increasing a_0 which we can ensure by increasing R_0.

By a comparison argument and the triangle inequality there are curves $\gamma_i \subset \partial\mathcal{B}_{2R_3} \cap \tilde{S}_i$ with $Length(\gamma_i) = 2mR_3$ and so that if $y \in S_i = S_{\beta R_3}(\gamma_i)$ and $r_y = d^\Sigma(0, y)$ then $\mathcal{B}_{r_y/2}(y) \subset \tilde{S}_i$. Hence by Corollary 5.5 one has the scaling invariant point wise estimate

$$\sup_{\mathcal{B}_{r_y/4}} (y)|A|^2 \leq C_3 r_y^{-2}. \tag{4.1}$$

where $C_3 = C_3(\beta, m)$ is uniform independent of C. By the Gauss equations, the uniform total curvature estimate and the Gauss-Bonnet theorem together yield

$$\int_{\gamma_i} k_g \leq 2\pi + 2C_2 m < C_1 m.$$

where we take $C_1 = 2C_2 + 2$.

We claim that there is an $\alpha > 0$ so that if for some $1 \leq i_1 < i_2 \leq n$ the Hausdorff distance δ_{i_1,i_2} between γ_{i_1} and γ_{i_2} is satisfies $\delta_{i_1,i_2} < \alpha R_3$ then

$$\left\{z + u(z)\mathbf{n}(z) : z \in \mathcal{T}_{R_3/4}(S_{i_1})\right\} \subset \cup_{y \in S_{i_2}} \mathcal{B}_{r_y/4}(y)$$

for a function $u \neq 0$ with

$$|\nabla u| + |A||u| \leq C_0' R_3^{-1} \delta_{i_1,i_2}.$$

Due to the scaling invariant pointwise curvature estimate, the claim essentially follows form a compactness argument and a rescaling in the same way as Corollary 5.10. By possibly shrinking α, Proposition 5.7 gives that any S_{i_1} that satisfies the condition is $\frac{1}{2}$-stable.

The argument is finished by showing that–provided C is large enough–there are a pair of sectors S_{i_1} and S_{i_2} that satisfy the claimed closeness for the given α. Indeed, by taking a_0 large enough we can take n as large as we like. By noting that all the γ_i lie in $B_{2R_3}(0)$ if we consider the homothetic scaling of the curves $\frac{1}{R_3}\gamma_i$ then by a diagonalization argument and subsequential convergence, for n sufficiently large there must be $\frac{1}{R_3}\gamma_{i_1}$ and $\frac{1}{R_3}\gamma_{i_2}$ that have Hausdorff distance less than α. We now choose a_0 to obtain this large n and so also get R_0 and R_2. The Lemma is proved by taking $C \geq R_2$, $R = 2R_3$ and $\gamma = \gamma_{i_1}$. □

Given a $\frac{1}{2}$-stable surface we expect it to have pointwise scale invariant curvature bounds and so expect it to contain a a large graph. There is no reason to believe such a graph will close up so it is likely a multi-valued graph. One wishes to make such a principle quantitative and that is the content of the following lemma (which Corollary II.1.34 of [**3**]):

Lemma 5.21. *Given $\omega > 8, 1 > \epsilon > 0, C_0$ and N there exist m and Ω so that the following holds:*

If $0 \in \Sigma$ is an embedded minimal disk containing a curve $\gamma \subset \partial\mathcal{B}_{r_1}$ with

(1) $\int_\gamma k_g \leq C_0 m$

(2) $Length(\gamma) = mr_1$

(3) $\mathcal{T}_{r_1/8}(S_{\Omega_1^2 \omega r_1}(\gamma))$ *is* $\frac{1}{2}$*-stable,*

then (after a rotation of $\mathbb{R}^3$*)* $S_{\Omega_1^2 \omega r_1}(\gamma)$ *contains an* N*-valued graph* Σ_N *over* $D_{\omega \Omega_1 r_1} \setminus D_{\Omega_1 r_1}$ *with gradient bounded by* ϵ *and so* $|A| \leq \epsilon / r$ *and* $\text{dist}_{S_{\Omega_1^2 \omega r_1}(\gamma)}(\gamma, \Sigma_N) < 4\Omega_1 r_1$.

Combining the preceding two lemmas gives Theorem 5.1.

CHAPTER 6

Estimates off the Axis

We now wish to show that if a multi-valued graph appears within a minimal disk far from the boundary of the disk then one can extend the graph within the minimal disk nearly all the way to the boundary. In contrast with the results of the preceding chapter this is a *global* fact about embedded minimal disks.

The main result in this direction is the following (which is Theorem 0.3 of [**3**]):

THEOREM 6.1. *Given $\tau > 0$ there exist $N, \Omega, \epsilon > 0$ so that the following hold: Let $\Sigma \subset B_{R_0}$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_{R_0}$. If $\Omega r_0 < 1 < R_0/\Omega$ and Σ contains an N-valued graph Σ_g over $D_1 \backslash D_{r_0}$ with gradient $\leq \epsilon$ and so that*

$$\Sigma_g \subset \left\{x_3^2 \leq \epsilon^2 \left(x_1^2 + x_2^2\right)\right\}$$

then Σ contains a 2-valued graph Σ_d over $D_{R_0/\Omega} \backslash D_{r_0}$ with gradient $\leq \tau$ and $(\Sigma_g)^M \subset \Sigma_d$

REMARK 6.2. Here Σ_g^M denotes the "middle" two sheets of Σ_g.

By Theorem 5.1, for a C blow-up pair (p, s) with C sufficiently large one obtains the small multi-valued graph near p as needed by the theorem. Furthermore, by Lemma 2.28 as long as the curvature is sufficiently large in a given ball there must be a C blow-up pair in that ball. Combining these two facts gives

THEOREM 6.3. *Given $N \in \mathbb{Z}^+$ and $\epsilon > 0$ there exist $C_1, C_2 > 0$ so that the following holds: Let $0 \in \Sigma \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_R$. If for some $R > r_0 > 0$ one has*

$$\sup_{B_{r_0} \cap \Sigma} |A|^2 \geq 4C_1^2 r_0^{-2},$$

then there exists (after a rotation of $\mathbb{R}^3$) an N-valued graph Σ_g over $D_{R/C_2} \backslash D_{2r_0}$ with gradient $\leq \epsilon$ and contained in $\Sigma \cap \left\{x_3^2 \leq \epsilon^2(x_1^2 + x_2^2)\right\}$.

The proof of the extension property–i.e. Theorem 6.1–consists of three main steps.

(1) First, one studies properties of large stable regions in a minimal surface. In particular, one wishes to show that large stable sectors (in the sense of the previous chapter) that are known to spiral many times on a small scale must contain large multi-valued graphs as in Theorem 6.1. One of the key difficulties is to ensure that one doesn't "lose sheets" as one goes outward.
(2) The second step is to prove a certain type of curvature estimate for an embedded minimal disk that lies in a thin slab. Roughly, speaking one shows that if a component Σ_3 of the disk lies between two other components, Σ_1 and Σ_2 which are connected to one another by a long curve that

passes through an "axis" then Σ_3 has bounded curvature and is in fact very flat. This essentially says that for an embedded minimal disk in a thin slab the axis of the disk must be localized. With further effort one removes the assumption about lying in a thin slab.

(3) The third step is to solve a plateau problem using the embedded minimal disk Σ as a barrier. In particular, the existence of a local multi-graph allows one to find a curve on Σ that spirals many times. By connecting this curve to $\partial\Sigma$ and choosing an appropriate arc in $\partial\Sigma$ one obtains a closed curve γ in Σ that is linked (in an appropriate sense) with the small multi-graph. By using Σ as a barrier one may appeal to [**?**] in order to find a stable embedded minimal disk Γ with $\partial\Gamma = \gamma$. By construction Γ will be spiralling through the sheets of the small multi-graph in Σ and so by the first step will contain a large multi-valued graph extending nearly to $\partial\Sigma$. Since the separation between sheets of this graph grows slowly as long as there are enough sheets, one finds components of Σ trapped between the sheets of Γ that satisfy the conditions of Step two. This implies that the surface Σ between the sheets of Γ is essentially stable, and so again by Step one contains the claimed multi-graph.

This is one of the most technically demanding parts of Colding and Minicozzi's argument, so we will focus only on the second step–as it is the most straightforward and possibly of independent interest.

1. Extending Multi-valued Graphs in Stable Surfaces

For the sake of completeness we include a statement of the main theorem regarding the extension of multi-graphs inside of *stable* embedded minimal disks. This is Theorem II.0.21 of [**3**]

THEOREM 6.4. *Given $\tau > 0$, there exist $N_1, \Omega_1, \epsilon > 0$ such that the following holds: Let $\Sigma \subset B_{R_0}$ be a stable embedded minimal disk with $\partial\Sigma \subset B_{r_0} \cup \partial B_{R_0} \cup \{x_1 = 0\}$ where $\partial\Sigma \backslash \partial B_{R_0}$ is connected. If $\Omega_1 r_0 < 1 < R_0/\Omega_1$ and Σ contains an N_1-valued graph Σ_g over $D_1 \backslash D_{r_0}$ with gradient $\leq \epsilon$,*

$$\Pi^{-1}(D_{r_0}) \cap \Sigma^M \subset \{|x_3| \leq \epsilon r_0\},$$

and a curve η connects Σ_g to $\partial\Sigma\backslash\partial B_{R_0}$ where

$$\eta \subset \Pi^{-1}(D_{r_0}) \cap \Sigma \backslash \partial B_{R_0}$$

then Σ contains a 2-valued graph Σ_d over $D_{R_0/\Omega_1} \backslash D_{r_0}$ with gradient $\leq \tau$.

REMARK 6.5. We note that the embeddedness of the stable surface seems to be needed in the proof.

2. Estimate between the sheets

The main technical estimate of [**3**] is the to show that if a piece of a embedded minimal disk lies between two other pieces of the disk that are connected only via a distant "axis" then the first piece is rather flat. As usual this is very plausible based on the helicoid, but given the lack of a priori structure for an arbitrary embedded minimal disk it is hard to even state precisely.

Before stating the result we need some terminology. Given points $p, q \in \mathbb{R}^3$ we denote the line segment connecting them by $\gamma_{p,q}$. For a given curve γ we say

γ is h-monotone (for $h > 0$) provided for each $y \in \gamma$ that $B_{4h}(y)$ has only one component which meets $B_{2h}(y)$. Qualitatively, this means that the curve tends not to come pack to y after passing through it (hence the name) we refer to Lemma 6.16 for more discussion of the geometric meaning of the definition. Theorem I.0.8 of [**3**] then states:

THEOREM 6.6. *There exist $c_1 \geq 4$ and $2c_2 < c_4 < c_3 \leq 1$ so that the following holds: Let $\Sigma \subset B_{c_1 r_0}$ be an embedded minimal disk with $\partial\Sigma \subset B_{c_1 r_0}$ and $y \in \partial B_{2r_0}$. Suppose that Σ_1, Σ_2 and Σ_3 are distinct compoeenents of $B_{r_0}(y) \cap \Sigma$ and*

$$\gamma \subset (B_{r_0} \cup T_{c_2 r_0}(\gamma_{0,y})) \cap \Sigma$$

is a curve with $\partial\gamma = \{y_1, y_2\}$ where $y_i \in B_{c_2 r_0}(y) \cap \Sigma_i$ and each component of $\gamma \backslash B_{r_0}$ is a $c_2 r_0$-almost monotone. If Σ_3' is a component of $B_{c_3 r_0}(y) \cap \Sigma_3$ with y_1, y_2 in distinct components of $B_{c_4 r_0}(y) \backslash \Sigma_3'$ then Σ_3' is a graph.

We prove this theorem under the additional hypothesis that Σ lies in a thin slab. This simplifies some of the arguments and is a key step in the proof of the general theorem.

3. The problem of Plateau and a theorem of Meeks and Yau

We briefly discuss here some existence results for minimal surfaces. Recall the classical problem of Plateau asks: Given a closed simple curve γ in $\mathbb{R}^3$ does there exist a minimal surface Γ with $\partial\Gamma = \gamma$? In this form the question was settled in the affirmative by Jesse Douglas and (independently) by Tibor Rado in the early 30s. A more refined version of the Plateau problem seeks, in addition to establishing the existence of some minimal Γ, to address the following issues:

(1) The regularity of Γ;
(2) The topological type of Γ;
(3) Whether the surface Γ is embedded.

In general these points are understood in terms of restrictions on the curve γ.

REMARK 6.7. For the record we note that the Douglas-Rado solution is automatically of disk type though it may fail to be embedded. Furthermore, in the original proofs the possibility of (interior) branch points could not be ruled out– their absence was later shown by Osserman (with subsequent work further clarifying the regularity at the boundary). In other words, the Douglas-Rado solution gives for any closed simple curve a smoothly immersed minimal disk Γ with $\partial\Gamma = \gamma$. Moreover, this solution has the least area amongst all branched immersions with boundary γ.

We cannot hope to survey all possible approaches to this question. Instead, we will record for future use a condition that will ensure that we can find an embedded stable solution to the Plateau problem. The embeddedness seems to be necessary in Colding and Minicozzi's proof.

Recall that we say a C^2 domain $\Omega \subset \mathbb{R}^3$ is *mean convex* provided the mean curvature of $\partial\Omega$ with respect to the outward normal is non-negative. It will be helpful to have a notion of mean convexity for less regular domains. To that end we follow [**?**] and make the following definition:

DEFINITION 6.8. Let Ω be a domain in $\mathbb{R}^3$ such that $\partial\Omega = \Sigma_1 \cup \ldots \cup \Sigma_k$ where each Σ_i satisfies:

(1) Each Σ_i is a compact subset of a C^2 surface $\hat{\Sigma}_i$ of $\mathbb{R}^3$ with $\hat{\Sigma}_i$ disjoint from Ω.
(2) Each Σ_i is the closure (in $\hat{\Sigma}_i$) of a domain $\mathring{\Sigma}_i$ which is a C^2 surface whose mean curvature with respect to the outward normal of Ω is non-negative.

then we say Ω is *(generalized) mean convex.*

As an example: if Σ is an embedded minimal surface in $\mathbb{R}^3$ with $\partial\Sigma \subset \partial B_R$ then $B_R \backslash \Sigma$ consists of two domains Ω_1 and Ω_2 both of which are mean convex in this sense.

We then have the following theorem summarizing several results of [**?**]:

THEOREM 6.9. *Let Ω be a mean convex region and suppose that γ is a closed simple curve in $\partial\Omega$. If γ is null-homotopic in $\overline{\Omega}$ then there is an embedded minimal disk $\Gamma \subset \overline{\Omega}$ with $\partial\Gamma = \gamma$ with least area amongst all immersed disks $\Gamma' \subset \Omega$ with $\partial\Gamma' = \gamma$. Moreover, either $\Gamma \subset \partial\Omega$ or $\overline{\Gamma} \cap \partial\Omega = \gamma$.*

REMARK 6.10. We remark that Γ is a smooth stable surface in its interior even if $\Gamma \subset \partial\Omega$ which can be useful if one wants to ensure that Γ lies in the interior of Ω.

4. Catenoid Foliation

In proving Theorem 6.6 it will be useful to know that a minimal surface lying inside of a thin slab cannot have large "holes". In particular, this will mean that distant points in such a surface Σ which can be joined by line segment that is "far" from $\partial\Sigma$ can actually be joined by curve in Σ that is close to the line segment. This is proved using catenoidal barriers.

Recall the standard vertical catenoid is given by

$$Cat = \left\{ x \in \mathbb{R}^3 : \cosh^2 x_3 = x_1^2 + x_2^2 \right\}.$$

For a $y \in \mathbb{R}^3$ we write $Cat(y) = Cat + y$ the vertical catenoid centered at y. Consider cones C_θ through the origin that are rotationally symmetric about the x_3 axis:

$$C_\theta = \left\{ x : x_3^2 = |x|^2 \sin^2 \theta \right\}.$$

One verifies using the convexity of $\cosh x_3$ that there is a unique such cone $C_0 = C_{\theta_0}$ that meets Cat tangentially. Denote by Cat_0 the unique bounded component of $Cat \backslash C_0$. Denote by Ω_0 the open cone given as the component of $\mathbb{R}^3 \backslash C_0$ containing Cat_0. Evidently, $\{\lambda Cat_0\}_{\lambda > 0}$ provides a foliation of Ω_0. More generally, $\{\lambda Cat_0 + y\}_{\lambda > 0}$ provides a foliation of $\Omega_0(y) = \Omega_0 + y$. We call this foliation the *catenoid foliation about y*. From the geometry of Cat and the cone C_0 there is a constant $\beta_A > 0$ so that away form a small ball the a slab of thickness β_A lies in Ω_0. Precisely, for a given y:

$$\{x : |x_3 - y_3| \leq 2\beta_A h\} \backslash B_{h/8}(y) \subset \Omega_0(y). \tag{4.1}$$

Moreover, the slab meets a suitable leaf of the foliation of Ω_0. Specifically,

$$(\frac{h}{16} Cat_0 + y) \cap \{x : |x_3 - y_3| \leq 2\beta_A h\} \subset B_{7h/32}(y). \tag{4.2}$$

Where here we note the constant β_A is uniform due to scaling and translation invariance. Notice that the catenoidal foliation about y restricts to a foliation of $\{x : |x_3 - y_3| \leq 2\beta_A h\} \backslash B_{h/8}(y)$ for each h.

By the strict maximum principal if Σ is a minimal surface with $\partial\Sigma \subset C_0$ and $(\lambda Cat_0 + y) \cap \Sigma = \emptyset$ for $\lambda > \lambda_0$ while $(\lambda Cat_0 + y) \cap \Sigma \neq \emptyset$ for $\lambda_0 - \epsilon < \lambda < \lambda_0$ then $(\lambda_0 \overline{Cat_0} + y) \cap \overline{\Sigma} \subset \partial\Sigma$ and likewise for the analogous result when $\lambda_0 < \lambda < \lambda_0 + \epsilon$. We deduce some consequences of this

LEMMA 6.11. *If* $\partial\Sigma \subset \partial B_h(y)$, $B_{3h/4}(y) \cap \Sigma \neq \emptyset$, *and*

$$\Sigma \subset B_h(y) \cap \{x : |x_3 - y_3| \leq 2\beta_A h\}$$

then $B_{h/4}(y) \cap \Sigma \neq \emptyset$.

PROOF. Scaling (4.2) by 4 and noting that

$$\{x : |x_3 - y_3| \leq 2\beta_A h\} \subset \{x : |x_3 - y_3| \leq 8\beta_A h\}$$

and $Cat_0 \cap B_1 = \emptyset$ gives that

$$\Sigma \cap (\frac{3h}{4} Cat_0 + y) \subset B_{7h/8} \backslash B_{3h/4}(y).$$

Hence by the strict maximum principle and (4.1) $B_{h/8}(y) \cap \Sigma$ is non-empty. □

Iterating this fact shows one can connect points in a minimal surface in a thin slab by an "approximate" line segment

COROLLARY 6.12. *If* $\Sigma \subset \{x : |x_3| \leq 2\beta_A h\}$ *is minimal and there are points* $p, q \in \{x_3 = 0\}$ *so that* $T_h(\gamma_{p,q}) \cap \partial\Sigma = \emptyset$ *and*

$$y_p \in B_{h/4}(p) \cap \Sigma$$

then there is a curve $\nu \subset T_h(\gamma_{p,q}) \cap \Sigma$ *that connects* y_p *to* $B_{h/4}(q) \cap \Sigma$.

REMARK 6.13. The curve constructed will be h-monotone in the sense discussed above.

PROOF. Choose points $y_0 = p, y_1, \ldots, y_n = q \in \gamma_{p,q}$ with $|y_{i-1} - y_i| = h/2$ for $i < n$ and $|y_{n-1} - y_n| \leq h/2$. Lemma 6.11 gives for each $1 \leq i \leq n$ a curve $\nu_i : [0,1] \to B_h(y_i) \cap \Sigma$ so that $\nu_1(0) = y_p$, $\nu_i(1) \in B_{h/4}(y_i) \cap \Sigma$ and $\nu_{i+1}(0) = \nu_i(1)$. We concatenate these curves to get ν. □

Finally, we prove a weak Rado-type theorem for minimal surfaces in a thin slab.

COROLLARY 6.14. *If* $\Sigma \subset \{x : |x_3| \leq 2\beta_A h\}$ *is a compact minimal surface and* E *is an unbounded component of* $\mathbb{R}^2 \backslash T_{h/4}(\Pi(\partial\Sigma))$ *then* $\Pi(\Sigma) \cap E = \emptyset$.

PROOF. Given $y \in E$ choose a curve $\gamma : [0,1] \to \mathbb{R}^2 \backslash T_{h/4}(\Pi(\partial\Sigma))$ with $|y(0)| > \sup_{x \in \Sigma} |x| + h$ and $\gamma(1) = y$. Set $\Sigma_t = \Sigma \cap (\frac{3h}{16} Cat_0 + \gamma(t))$. By (4.2), $\Sigma_t \subset B_{7h/32}(\gamma(t))$ so that $\Sigma_0 = \emptyset$ and $\Sigma_t \cap \partial\Sigma = \emptyset$. Since $\Sigma_0 = \emptyset$ and $\partial\Sigma \cap (\frac{3h}{16} Cat_0 + \gamma(t)) = \emptyset$ and $\Sigma \cap \partial(\frac{3h}{16} Cat_0 + \gamma(t)) = \emptyset$ the strict maximum principle implies that there is not a first $t > 0$ so that $\Sigma_t \neq \emptyset$. □

5. Thin Slab

We now want to show Theorem 6.6, the estimate between the sheets, for minimal disks that lie within a thin slab. By working in a thin slab we will be able to make use of the results of the preceding section regarding the lack of big "holes" in minimal surfaces as well as the following a priori bound for stable surfaces

LEMMA 6.15. *Let $\Gamma \subset \{x : |x_3| \leq \beta h\}$ be a stable embedded minimal surface. There exist $C_g, \beta_s > 0$ so that if $\beta \leq \beta_s$ and E is a component of*

$$\mathbb{R}^2 \backslash T_h(\Pi(\partial\Gamma)),$$

then each component of $\Pi^{-1}(E) \cap \Gamma$ is a graph over E of a function u with $|\nabla u| \leq C_g \beta$.

PROOF. We may scale so $h = 1$. If $\mathcal{B}_1(y) \subset \Gamma \backslash \partial\Gamma$ the curvature estimate of Theorem 2.21 gives a uniform curvature estimate on $\mathcal{B}_{1/2}(y)$. This allows us to argue by taking $\beta \to 0$ and seeing that $\mathcal{B}_{1/2}(y)$ would converge smoothly and with multiplicity one to disk $D_{1/2}(\Pi(y))$. In other words, for β_s sufficiently small each $\mathcal{B}_{1/2}(y)$ can be written as a graph of a function u with small gradient. It is straightforward to patch these graphs together and prove the lemma. □

We also need the following lemma that helps to explain the motivation for h-montonicity:

LEMMA 6.16. *Fix a point $y \in \partial B_{2Ch}(0)$ where $C > 3$. Suppose that γ is a connected h-monotone curve with $\gamma \subset T_h(\gamma_{0,y}) \cup B_{2h}(0) \cup B_{2h}(y)$ that connects $B_{2h}(0)$ to $B_{2h}(y)$. If $\partial B_{Ch}(0)$ meets γ transversally then $\partial B_{Ch}(0) \cap \gamma$ consists of an odd number of points.*

REMARK 6.17. Roughly speaking this lemma tells us that if a h-monotone curve is Hausdorff close to a line segment then it should be so with "multiplicity" one. As there is very little geometric control on the curve this should be understood as a "topological" multiplicity. For instance an example of such a curve that was not h-monotone would be if γ was obtained from a closed loop by cutting out a small sector in $B_{2h}(0)$.

PROOF. There must be at least one point $x \in \gamma \cap B_{Ch}$ otherwise γ wouldn't be connected. Consider the component γ' of $B_{4h}(x) \cap \gamma$ that contains x. By the h-monotonicity $\gamma' \cap B_{2h}(x) = \gamma \cap B_{2h}(x)$. In particular, the plane orthogonal to $\gamma_{0,y}$ through x separates $\partial\gamma'$. The proof now follows by a counting argument, namely we count the number of intersections of $\partial B_t(0) \cap \gamma'$ at regular values of t. This number is locally constant and jumps by 1 at a point of $\partial\gamma'$ and by an even number at any point of non-transverse intersection. For $t = h$, $\partial B_t(0) \cap \gamma'$ is empty □

We next use the catenoid foliation to show that if one has a non-graphical point near the center of an embedded minimal disk Σ in a thin slab, then it is possible to find a curve γ coming near the center that connects two points on $\partial\Sigma$ that are close extrinsically but not intrinsically.

LEMMA 6.18. *Let $\Sigma \subset B_{60h} \cap \{x : |x_3| \leq \beta_A h\}$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_{60h}$ and let $z_b \in \partial B_{50h}$. If a component Σ' of $B_{5h} \cap \Sigma$ is not a graph then there are:*

1. *Distinct components S_1, S_2 of $B_{8h}(z_b) \cap \Sigma$.*
2. *Points z_1 and z_2 with $z_i \in B_{h/4}(z_b) \cap S_i$.*
3. *A curve $\gamma \subset B_{30h} \cup T_h(\gamma_{q,z_b}) \cap \Sigma$ with $\partial\gamma = \{z_1, z_2\}$ and $\gamma \cap \Sigma' \neq \emptyset$. Here $q \in B_{50h}(z_b) \cap \partial B_{30h}$.*

PROOF. The proof of this fact has three main steps

1. We first build a curve γ_a near the non-graphical point roughly on scale h. This will use the catenoidal foliation.

(2) The second step is to use the maximum principle to restrict what types of curves can connect the endpoints of γ_a. Essentially, one uses the maximum principle to show that there is a half-space H a fixed distance from the non-graphical point that contains the endpoints of γ_a and so that it is impossible to join the endpoints by a connected curve in $\Sigma \cap H$.
(3) The final step is to extend γ_a in $\Sigma \cap H$ to get the desired γ. The properties of γ will follow from the lack of connecting curve deduced in the second step.

For the sake of time we discuss only the first two steps. As Σ' is not graphical, there is a point $z \in \Sigma'$ so that $\mathbf{n}(z) \cdot \mathbf{e}_3 = 0$. Fix a point $y \in \partial B_{4h}(z)$ so that $\gamma_{y,z}$ is normal to Σ at z. Pick $y' \in \partial B_{10h}(y)$ so that $z \in \gamma_{y,y'}$. Our goal is to show that up to swapping y and y' that we can find a curve γ_a satisfying:

(1) $\gamma_a \subset (T_h(\gamma_{y,y'}) \cup B_{5h}(y')) \cap \Sigma$
(2) $\partial\gamma_a = \{y_1, y_2\} \subset B_{h/4}(y)$
(3) Each $y_i \in S_i^a$ for components $S_1^a \neq S_2^a$.

To see this we let $A_h(y) = \cup_{\lambda > 4h}(\lambda Cat_0 + y)$ be an annular region in $\Omega_0(y)$. Any simple closed curve $\sigma \subset \Sigma \backslash A_h(y)$ bounds a disk $\Sigma_\sigma \subset \Sigma$. By the strict maximum principal, $\Sigma_\sigma \cap A_h(y) = \emptyset$. Notice that $z \in \partial A_h(y)$ and that by construction Σ and $\partial A_h(y)$ meet tangential at z. In particular, there is a (small) neighborhood of U_z of z in Σ so that $U_z \cap \partial A_h(y) \backslash z$ is the union of $2n \geq 4$ disjoint embedded arcs meeting at z. Furthermore, $U_z \backslash A_h(y)$ consists of n components $U_1, \ldots, U_n$ with $\overline{U}_i \cap \overline{U}_j = \{z\}$ for $i \neq j$. To see this we remark that near z both Σ and $\partial A_h(y)$ can be written as graphs over the plane $P = T_z\Sigma = T_z\partial A_h(y)$. Since near z both graphs will be approximately the graph of a harmonic function this description follows from analogous results for level sets of harmonic functions.

We claim that it is impossible for a simple curve $\tilde{\sigma}_z \subset \Sigma \backslash \overline{A_h(y)}$ to connect U_1 to U_2. In other words, there are distinct components Σ_{4h}^1 and Σ_{4h}^2 of $\Sigma \backslash \overline{A_h(y)}$ with $U_i \subset \Sigma_{4h}^i$. That is, we cannot connect the small disjoint sets U_i lying outside the annulus $A_h(y)$ by a curve also outside the annulus. Indeed, suppose there was such a curve. By connecting $\partial\tilde{\sigma}_z$ by a curve in U_z one obtains a simple closed curve $\sigma_z \subset \Sigma \backslash A_h(y)$ containing $\tilde{\sigma}_z$ and with $\sigma_z \cap \overline{A_h(y)} = \{z\}$. As a consequence, σ_z bounds a (open) disk $\Sigma_s \subset \Sigma \backslash A_h(y)$. Indeed, take $r > 0$ small enough so that $\mathcal{B}_r(z) \subset U_z$ Then since Σ_s has boundary σ_z, $\mathcal{B}_r \cap \Sigma_s$ should be connected, but by construction $U_z \cap \Sigma_s$ is not–giving the desired contradiction.

Notice that $z \in \overline{\Sigma}_{4h}^1 \cap \overline{\Sigma}_{4h}^2$. By (4.2), there are $y_i^a \in B_{h/4}(y) \cap \Sigma_{4h}^i$. Corollary 6.12 then gives curves $\nu_i \in T_h(\gamma_{y,y'}) \cap \Sigma$ with $\partial\nu_i = \left\{y_i^a, y_i^b\right\}$ where $y_i^b \in B_{h/4}(y')$. There are now two possibilities

(1) There is a $\hat{\gamma}_0 \subset B_{4h}(y') \cap \Sigma$ that connects y_1^b to y_2^b.
(2) y_1^b and y_2^b do not connect in $B_{4h}(y') \cap \Sigma$.

In the first case we take $\gamma_a = \nu_1 \cup \hat{\gamma}_0 \cup \nu_2$ and $y_i = y_i^a$. In the second case take $\gamma_0 \subset B_{5h}(y) \cap \Sigma$ connecting y_1^a to y_2^a (such a curve exists by connecting y_i^a to z). Then set $\gamma_a = \nu_1 \cup \hat{\gamma}_0 \cup \nu_2$ and take $y_i = y_i^b$. In the first case we are done while in the second we must swap y and y'. We take S_i^a to be the (distinct) components of $B_{4h}(y) \cap \Sigma$ with $y_i \in S_i^a$.

To show the next step we must rule out the existence of certain curves from y_1 to y_2. Let $H = \{x : \langle y - y', x - y\rangle > 0\}$. So $y' \notin H$. If there is a curve $\eta_{1,2} \subset T_h(H) \cap \Sigma$ connecting y_1 and y_2 then $\eta_{1,2} \cup \gamma_a$ bounds a disk $\Sigma_{1,2} \subset \Sigma$ by the maximum principle. Since $\eta_{1,2} \subset T_h(H)$

$$\partial B_{8h}(y') \cap \partial\Sigma_{1,2} = \partial B_{8h}(y') \cap \gamma_a = \partial B_{8h}(y') \cap (\nu_1 \cup \nu_2)\,. \tag{5.1}$$

Arguing as in Lemma 6.16, this implies that $\partial B_{8h}(y') \cap \partial\Sigma_{1,2}$ consists of an odd number of points in each S_i^a. In hence, there is a curve in $\partial B_{8h}(y') \cap \Sigma_{1,2}$ connecting S_1^a to S_2^a. As S_1^a and S_2^a are distinct components of $B_{4h}(y) \cap \Sigma$ and this curve must contain a point

$$y_{1,2} \in \partial B_{4h}(y) \cap \partial B_{8h}(y') \cap \Sigma_{1,2}.$$

Since $y_{1,2} \in \partial B_{4h}(y) \cap \partial B_{8h}(y') \cap \{|x_3| \leq \beta_A h\}$, an elementary geometric argument gives that $\Pi(y_{1,2}) \notin \Pi(T_{h/4}(\gamma_{y,y'}))$. In other words, $\Pi(y_{1,2})$ is in an unbounded component of $\mathbb{R}^2 \setminus T_{h/4}(\Pi(\partial\Sigma_{1,2}))$, which contradicts Corollary 6.14. This contradiction shows that y_1 and y_2 cannot be connected in $T_h(H) \cap \Sigma$. □

We are now ready to prove Theorem 6.6 for disks in thin slabs. To that end set

$$\beta_3 = \min\{\beta_A, \beta_s, \tan\theta_0/(2C_g)\}$$

where here β_A and θ_0 are defined in Section 4 and C_g and β_s are from Proposition 6.15. This will be our measure of "thinness".

We then have

PROPOSITION 6.19. *Let $\Sigma \subset B_{4r_0} \cap \{|x_3| \leq \beta_3 h\}$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_{4r_0}$ and let $y \in \partial B_{2r_0}$. Suppose that $\Sigma_1, \Sigma_2, \Sigma_3$ are distinct components of $B_{r_0}(y) \cap \Sigma$ and*

$$\gamma \subset (B_{r_0} \cup T_h(\gamma_{0,y})) \cap \Sigma$$

is a curve with $\partial\gamma = \{y_1, y_2\}$ where $y_i \in B_h(y) \cap \Sigma_i$ and each component of $\gamma \setminus B_{r_0}$ is h-monotone.

If Σ_3' is a component of $B_{r_0-80h}(y) \cap \Sigma_3$ for which y_1, y_2 are in distinct components of $B_{5h}(y) \setminus \Sigma_3'$ then Σ_3' is a graph.

REMARK 6.20. Note that r_0 and h are essentially arbitrary though we will need to take $\frac{h}{r_0} << 1$.

PROOF. Let us summarize the proof.

(1) Assume Σ_3' is not graphical. Apply Lemma 6.18 to find a connected curve γ_3 connecting Σ_3' to $\partial\Sigma$ so that away from Σ_3' γ_3 consists of two approximate lines in distinct components.
(2) Next we find disjoint stable disks Γ_1, Γ_2 near y so that Σ_3' is sandwiched between Γ_1 and Γ_2. Moreover, $\Gamma_1 \cup \Gamma_2$ mutually separates y_1, y_2 and Σ_3'.
(3) The third step is to solve the Plateau problem (using Σ as a barrier) for a closed curve that is the union of γ_3 and an appropriate part of $\partial\Sigma$. This will yield a stable surface Γ_3. If the part of $\partial\Sigma$ is chosen appropriately Γ_3 will pass between γ.
(4) Finally, since Γ_3 must be very flat it will follow from the position of $\partial\Gamma_3$ that Γ_3 must intersect Σ a contradiction.

Suppose Σ_3' is not a graph. Fix a non-graphical point $z \in \Sigma_3'$. We define points z_0, y_0, y_b on the ray $\overline{0,y}$ (that is the ray starting at 0 and going through y) by

$$z_0 = \partial B_{3r_0-21h} \cap \overline{0,y}$$

$$y_0 = \partial B_{3r_0-10h} \cap \overline{0,y}$$

$$y_b = \partial B_{4r_0} \cap \overline{0,y}.$$

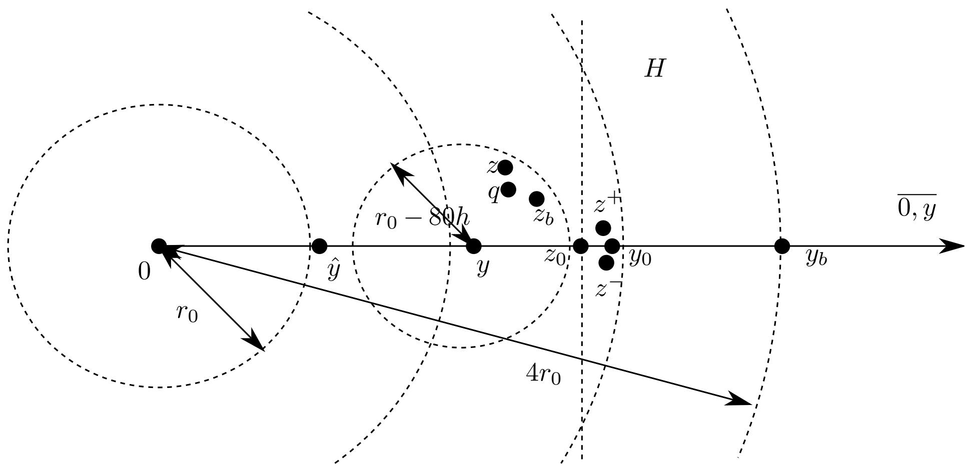

FIGURE 1. Various points in the proof

We set $z_b = \partial B_{50h}(z) \cap \gamma_{z,z_0}$ and define the half-space

$$H = \{x : \langle x - z_0, z_0 \rangle > 0\} .$$

So H is the region not containing 0 that is bounded by the plane orthogonal to $\overline{0, y}$ through z_0. Notice that $y_0, y_b \in H$ while $y, z_b \notin H$. We refer to Figure 3.

We claim that there is a simple curve γ_3 so that

(1)
$$\gamma_3 \subset (B_{r_0-20h}(y) \cup T_h(\gamma_{y,y_b})) \cap \Sigma,$$
(2) γ_3 can be connected to Σ_3' in $B_{r_0-20h}(y) \cap \Sigma$,
(3) $\partial\gamma_3 \subset \partial\Sigma$,
(4) There are two distinct components of $H \cap \Sigma$ in which $\partial B_{r_0-10h}(y) \cap \gamma_3$ consists of an odd number of points.

To find such γ_3 we first apply Lemma 6.18 to find a $q \in B_{50h}(z_b) \cap \partial B_{30h}(z)$, distinct components S_1, S_2 of $B_{8h}(z_b) \cap \Sigma$ with $z_i \in B_{h/4}(z_b) \cap S_i$ and a curve (see Figure 2)

$$\gamma_3^* \subset (B_{30h}(z) \cup T_h(\gamma_{q,z_b})) \cap \Sigma, \partial\gamma_3^* = \{z_1, z_2\}, \gamma_3^* \cap \Sigma_3' \neq \emptyset.$$

By Corollary 6.12 there are h-monotone curves

$$\nu_1, \nu_2 \subset T_h(\gamma_{z_b,z_0} \cup \gamma_{z_0,y_b}) \cap \Sigma$$

and so that ν_i connects z_i to $\partial\Sigma$ $(i = 1, 2)$. Then $\gamma_3 = \nu_1 \cup \gamma_3^* \cup \nu_2$ extends γ_3^* to $\partial\Sigma$. The curve γ_3 as constructed verifies items (1), (2) and (3). Now fix points $z^+ \in B_h(y_0) \cap \nu_1$ and $z^- \in B_h(y_0) \cap \nu_2$. We claim that z^-, z^+ cannot be connected in $H \cap \Sigma$. Indeed, if ν_+^- connects z^- to z^+ then ν_+^- together with the portion of γ_3 connecting z^- to z^+ and going near Σ_3' bounds disk $\Sigma_+^- \subset \Sigma$. By the h-monotonicity $\partial B_{50h}(z) \cap \partial\Sigma_+^-$ consists of an odd number of points in each S_i. Hence, there is a curve $\sigma_+^- \subset \partial B_{50h}(z) \cap \Sigma_+^-$ that connects S_1 to S_2 and so that $\sigma_+^- \setminus B_{8h}(z_b) \neq \emptyset$. This contradicts Corollary 6.14 and so we conclude that there are distinct components of Σ_H^+ and Σ_H^- of $H \cap \Sigma$ with $z^\pm \in \Sigma_H^\pm$. Final remove any loops in γ_3 to make it simple.

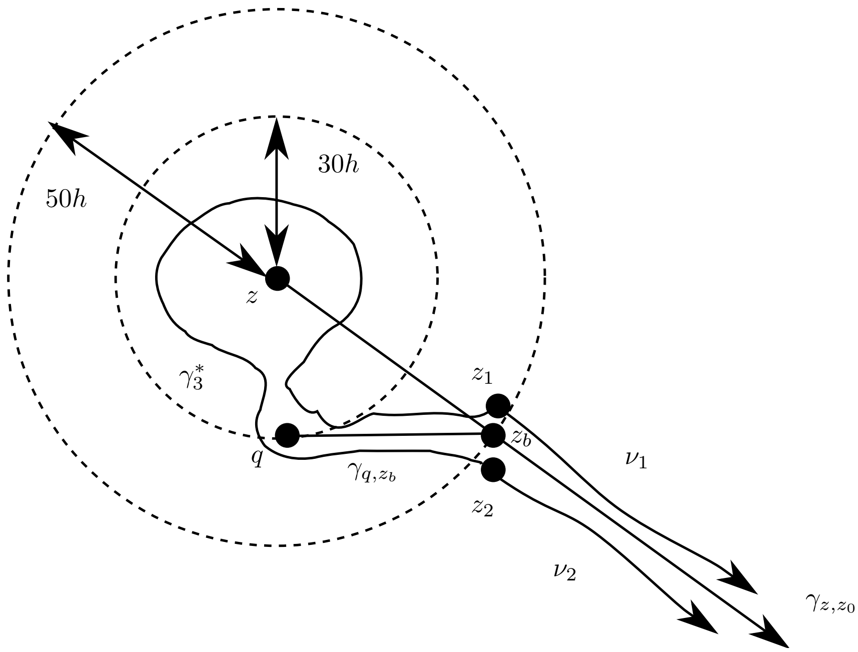

FIGURE 2. The curve γ_3^*

We next find disjoint stable disks

$$\Gamma_1, \Gamma_2 \subset B_{r_0-2h}(y) \backslash \Sigma$$

with $\partial\Gamma_i \subset \partial B_{r_0-2h}(y)$ and a graphical component Γ_i' of $B_{r_0-4h}(y) \cap \Gamma_i$ so that Σ_3' is between Γ_1', Γ_2' and y_1, y_2 and Σ_3' are all in their own component of

$$B_{r_0-4h}(y) \backslash (\Gamma_1' \cup \Gamma_2').$$

We achieve this by solving appropriate Plateau problems. Let Σ_1' and Σ_2' be the components of $\Sigma \cap B_{r_0-2h}(y) \cap \Sigma$ with $y_i \in \Sigma_i'$. Both are disks by the maximum principle. Denote by Σ_{y_2} the component of $B_{3h}(y_1) \cap \Sigma$ containing y_2. Since $y_1 \notin \Sigma_{y_2}$, one can use a catenoid foliation centered at y_1 to see that there is a $y_2' \in \Sigma_{y_2} \backslash \Omega_0(y_1)$. From this it follows that the vector $y_1 - y_2'$ is nearly orthogonal to the slab, that is

$$|\Pi(y_2' - y_1)| \leq |y_2' - y_1| \cos\theta_0.$$

Here θ_0 is the critical cone angle from Section 4. By assumption, Σ_3 separates y_1 and y_2 in $B_{5h}(y)$ and so $\gamma_{y_1,y_2'}$ is linked with $\partial\Sigma_3'$. Since Σ_3' is a disk this implies that there is a $y_3 \in \gamma_{y_1,y_2'} \cap \Sigma_3'$. Fix a component Ω_1 of $B_{r_0-2h}(y)\backslash\Sigma$ containing a component of γ_{y_1,y_3} with exactly one endpoint in Σ_1'. By Theorem 6.9, there is a stable embedded disk $\Gamma_1 \subset \overline{\Omega}_1$ with $\partial\Gamma_1 = \partial\Sigma_1'$ (we allow $\Gamma_1 = \Sigma_1$). Similarly, let Ω_2 be a component of $B_{r_0-2h}(y)\backslash(\Sigma \cup \Gamma_1)$ with exactly one endpoint of $\gamma_{y_3,y_2'}\backslash(\Sigma\cup\Gamma_1)$ in Σ_2'. Again one obtains a stable embedded disk $\Gamma_2 \subset \overline{\Omega}_2$ with $\partial\Omega_2 = \partial\Sigma_2'$. Note, Ω_2 Ω_1 are distinct as Σ_3' separates. allowed?] As $\partial\Gamma_1$ is linked in Ω_1 with γ_{y_1,y_3} and $\partial\Gamma_2$ is linked in Ω_2 with $\gamma_{y_3,y_2'}$ there are components Γ_i' of $B_{r_0-4h}(y)\cap\Gamma_i$ with $z_1^\gamma \in \Gamma_1' \cap \gamma_{y_1,y_3}$ and $z_2^\Gamma \in \Gamma_2' \cap \gamma_{y_3,y_2'}$. By Proposition 6.15, each

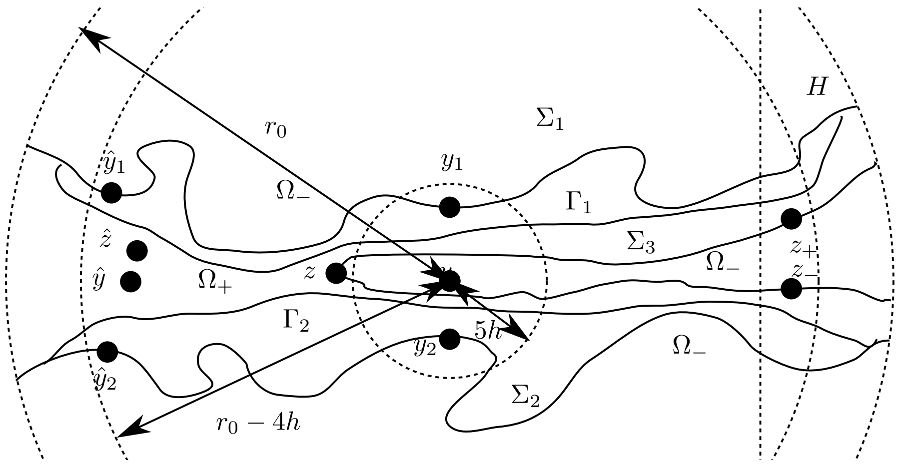

FIGURE 3. An illustration of a non-graphical Σ_3'

Γ_i' is a graph of a function u_i with $|\nabla u_i| \leq C_g \beta_3$. Our choice of β_3 ensures that

$$\Gamma_i' \setminus \left\{ z_i^\Gamma \right\} \subset \Omega_0(z_i^\Gamma). \tag{5.2}$$

Since $y_1 - y_2'$ is nearly orthogonal to the slab we have $\gamma_{y_1, y_2'} \cap \Omega_0(z_i^\Gamma) =$ and so $\Gamma_i' \cap \gamma_{y_1, y_2'} = \left\{ z_i^\Gamma \right\}$. In particular, we then see that y_1, y_2, y_3 are all in distinct components of

$$B_{r_0 - 4h}(y) \setminus (\Gamma_1' \cup \Gamma_2') \, .$$

We have completed the second step.

Set $\hat{y} = \partial B_{r_0 + 10h}(0) \cap \gamma_{0,y}$. so $\hat{y} \notin H$. Let $\hat{\gamma}$ be the component of $B_{r_0+10h} \cap \gamma$ with $B_{r_0} \cap \hat{\gamma} \neq \emptyset$. Then $\partial \hat{\gamma} = \{\hat{y}_1, \hat{y}_2\}$ with $\hat{y}_i \in B_h(\hat{y}) \cap \Sigma_i'$. This follows since γ lies in the tubular neighborhood $T_h(\gamma_{0,y})$.

We will now seek to solve a Plateau problem and find an embedded stable disk Γ_3 that separates $\hat{y}_1$ from $\hat{y}_2$. It may be helpful to refer to Figure **??**. To solve the Plateau problem, we note that the curve γ_3 divides Σ into two sub-disks $\Sigma_3^\pm$. We let $\Omega^\pm$ be the components of $B_{4r_0} \setminus (\Sigma \cup \Gamma_1 \cup \Gamma_2)$ with $\gamma_3 \subset \partial \Omega^+ \cap \partial \Omega^-$. Note that both $\Omega^\pm$ are mean convex in the sense of Meeks-Yau since $\partial \Gamma_1 \cup \partial \Gamma_2 \subset \Sigma$ and $\partial \Sigma \subset \partial B_{4r_0}$. From our first step, we may label $\Omega^\pm$ so that the $z^\pm$ do not connect in $H \cap \Omega^+$, indeed if we could connect z^+ to z^- in $H \cap \Omega^+$ and in $H \cap \Omega^-$ it would be possible to connect them in $H \cap \Sigma$ (basic idea let τ_1, τ_2 be connecting curves in $H \cap \Omega^+$ and $H \cap \Omega^-$–then there is a disk $D \subset H$ with $\partial D = \tau_1 \cup \tau_2$ and $D \cap \Sigma$ would connect z^- and z^+). By Theorem 6.9, we get a stable embedded disk $\Gamma_3 \subset \Omega^+$ with $\partial \Gamma_3 = \partial \Sigma_3^+$. By the h-monotonicity of γ_3 one has $\partial B_{r_0 - 10h}(y) \cap \partial \Gamma_3$ consists of an odd number of points in each of Σ_H^+, Σ_H^- (see Lemma 6.16). Hence, there is a curve,

$$\gamma_+^- \subset \partial B_{r_0 - 10h}(y) \cap \Gamma_3$$

from Σ_H^+ to Σ_H^-. By construction $\gamma_+^- \setminus B_{8h}(y_0) \neq \emptyset$ since otherwise z_+ and z_- would connect in $H \cap \Omega^+$. Hence, since

$$\partial B_{r_0 - 10h}(y) \cap T_h(\partial \Gamma_3) \subset B_{3h}(y_0),$$

Proposition 6.15 gives $\hat{z} \in B_h(\hat{y}_1) \cap \gamma_+^-$. By the second step, Γ_3 is between Γ_1' and Γ_2' .

Let $\hat{\Gamma}_3$ be the component of $B_{r_0+19h} \cap \Gamma_3$ with $\hat{z} \in \hat{\Gamma}_3$. By Proposition 6.15 $\hat{\Gamma}_3$ is a graph. Finally, since $\hat{\gamma} \subset B_{r_0+10h}$ and $\hat{\Gamma}_3$ passes between $\partial\hat{\gamma}$ this forces $\hat{\Gamma}_3$ to intersect $\hat{\gamma}$, which is impossible and proves the proposition.

□

CHAPTER 7

The One-sided Curvature Estimate

Bibliography

1. T. H. Colding and W. P. Minicozzi II, *Minimal surfaces*, Courant Lecture Notes in Math., New York University, Courant Institute of Math. Sciences, New York, 1999.
2. ______, *The space of embedded minimal surfaces of fixed genus in a 3-manifold II; multi-valued graphs in disks*, Ann. of Math. (2) **160** (2004), no. 1, 69–92.
3. ______, *The space of embedded minimal surfaces of fixed genus in a 3-manifold III; planar domains*, Ann. of Math. (2) **160** (2004), no. 2, 523–572.
4. ______, *The Calabi-Yau conjectures for embedded surfaces*, Ann. of Math. (2) **167** (2008), no. 1, 211–243.
5. W. H. Meeks III and H. Rosenberg, *The uniqueness of the helicoid*, Ann. of Math. (2) **161** (2005), no. 2, 727–758.