Problem #6. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be $C^1$. Suppose that the autonomous system $X' = F(X)$ admits a global solution $X(t)$ with $X(t) = C \cos t E_1 + C \sin t E_2$ for some $C > 0$, i.e. $X(t)$ parameterizes a circle.

(a) Show that if $n = 2$ and $|X_0| < C$, then the solution to the IVP

$$\begin{cases} X' = F(X) \\ X(0) = X_0 \end{cases}$$

must satisfy $|X(t)| < C$.

Solution. Most of you had the right idea on this one, but you must be rigorous! My solution is as follows:

Let $X(t)$ be the solution to the IVP above. Suppose that $|X(t_1)| \geq C$ for some $t_1$ (I’ll assume $t_1 > 0$ for simplicity; the $t_1 < 0$ case is almost identical). Then $|X(t)|$ is a continuous function, so there exists at $t_2 \in (0, t_1)$ such that $|X(t_2)| = C$ by the intermediate value theorem.

Now, let $S = \{t \in (0, t_2) : |X(t)| = C\}$, and let $t_3 = \inf S$ (the infimum used here is the greatest lower bound of this set). I claim that $t_3$ is actually in $S$. That is, I claim that $|X(t_3)| = C$. To see this, note that for any $\varepsilon > 0$, $[t_3, t_3 + \varepsilon) \cap S \neq \emptyset$ (this is a property of the infimum). Thus, there always exists some $t \in [t_3, t_3 + \varepsilon) \cap S$, and by construction, $|X(t)| = C$. Since $t$ is arbitrarily close to $t_3$ and $|X(t)|$ is continuous, it follows that $|X(t_3)| = C$ as claimed.

Now consider the IVP given by

$$\begin{cases} Y' = F(Y) \\ Y(t_3) = X(t_3) \end{cases}$$

(using our particular solution $X$ that we have been considering).

Now, the local uniqueness theorem says that there exists some $a > 0$ for which this IVP has a unique solution on the domain $(t_3 - a, t_3 + a)$. On the one hand, our $X(t)$ is clearly such a solution. On the other hand, we have the solution $X_{\text{circ}}(t) = C \cos t E_1 + C \sin t E_2$. This is not necessarily a solution of the above IVP, but we can change variables to make it so. After all, there exists some $t_4$ for which $X_{\text{circ}}(t_4) = X(t_3)$. Then we simply consider $Y(t) = X_{\text{circ}}(t + t_4 - t_3)$. By construction, $Y$ is another solution to the IVP, and $|Y(t)| = C$ for all $t$. But $|X(t_3 - a/2)| < C$ since $t_3$ is the infimum of $S$.

Thus, $X$ and $Y$ differ at the point $t_3 - a/2$, which is in $(t_3 - a, t_3 + a)$, which contradicts the local uniqueness theorem. Hence, we conclude that $|X(t)| < C$ for all $t$. \qed
Problem #7. Show that if $u : [a, b] \to \mathbb{R}$ is a $C^1$ function that satisfies the differential inequality

$$u' \leq \mu u + g(t),$$

where $g$ is continuous, then, for $t \in [a, b]$,

$$u(t) \leq u(a)e^{\mu(t-a)} + \int_a^t e^{\mu(t-s)}g(s)ds.$$

Solution. This follows the same method as solving a differential equation with an integrating factor. First, we move a term over.

$$u' - \mu u \leq g(t)$$

Next, we multiply both sides by $e^{-\mu t}$. This does not change the inequality sign because $e^{-\mu t}$ is always positive. We get

$$u'e^{-\mu t} - \mu ue^{-\mu t} = \frac{d}{dt}(ue^{-\mu t}) \leq g(t)e^{-\mu t}$$

The equality on the left is just the product rule. Now, there is a property of integrals that if $c \leq d$ and $f(t) \leq h(t)$ for all $t \in (a, b)$, then $\int_c^d f(t)dt \leq \int_c^d h(t)dt$. Applying this to the above, we get

$$\int_a^t \frac{d}{ds}(ue^{-\mu s})ds = u(t)e^{-\mu t} - u(a)e^{-\mu a} \leq \int_a^t g(s)e^{-\mu s}ds$$

for any $t \in [a, b]$. Adding a term back and multiplying by $e^{\mu t}$, which is positive for all $t$, we get

$$u(t) \leq u(a)e^{\mu(t-a)} + \int_a^t e^{\mu(t-s)}g(s)ds$$

as desired. \qed

Problem #8. Show that if $u : [a, b) \to \mathbb{R}$ is a positive $C^1$ function that satisfies the differential inequality

$$u' \geq \mu u^2$$

for $\mu > 0$, then we must have $b \leq a + \frac{1}{u(a)\mu}$.

Solution. Since $u^2 > 0$, we can divide by it without changing inequality signs. We get

$$\frac{u'}{u^2} \geq \mu$$

Integrate both sides from $a$ to $t \in [a, b)$ to get

$$\frac{1}{u(a)} - \frac{1}{u(t)} \geq \mu(t - a)$$
Note that since \( \frac{1}{u(t)} > 0 \), we have
\[
\frac{1}{u(a)} \geq \frac{1}{u(a)} - \frac{1}{u(t)} \geq \mu(t-a)
\]
for all \( t \in [a, b) \). Taking the limit as \( t \to b \), we get \( \frac{1}{u(a)} \geq \mu(b-a) \). Multiply each side by the positive quantity \( \frac{u(a)}{b-a} \) to obtain
\[
\frac{1}{b-a} \geq \frac{u(a)}{u(b-a)}
\]
Since both sides of the inequality are positive, inversion flips the inequality sign, hence
\[
b - a \leq \frac{1}{u(a)} \mu
\]
The desired result follows.

**Problem #9.** Use Problem #7, the Cauchy-Schwarz inequality, and the theorem on pg. 146 to show that if \( F : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^1 \) and satisfies \( |F(X)| \leq C|X| \) for some \( C > 0 \), then the IVP
\[
\begin{align*}
X' &= F(X) \\
X(0) &= X_0
\end{align*}
\]
has a global solution.

**Solution.** Let \( u(t) = |X(t)|^2 \). The first thing to notice is that
\[
u'(t) = \frac{d}{dt} |X(t)|^2 = \frac{d}{dt} (X(t) \cdot X(t)) = 2X'(t) \cdot X(t) = 2F(X) \cdot X \leq 2|F(X)| \cdot |X| \leq 2C|X|^2
\]
where we’ve used Cauchy-Schwarz at the last step. Using what we know about \( F \), this says that \( u' \leq 2C|X| \cdot |X| = 2Cu \).

Now, we know that there exists a maximal interval on which the solution \( X(t) \) exists. Call it \((\alpha, \beta)\), and suppose that \( \beta < \infty \). Consider any closed interval \([a, b] \subset (\alpha, \beta)\). By Problem #7, the above implies that
\[
u(t) \leq u(a)e^{2C(t-a)}
\]
As a consequence, \( |X(t)| \leq u(a)e^{C(t-a)} \) for all \( t \in [a, b] \). Since \( b \) could be anything in \((a, \beta)\), this indeed holds for all \( t \in [a, \beta) \). This implies that \( \lim_{t \to \beta} u(t) < \infty \), which contradicts the theorem on pg. 146. Hence, we conclude that \( \beta = \infty \). A similar argument shows that \( \alpha = -\infty \), hence the claim is proved.