Problem (1). We need to find a bijection $\phi$ between $P(A)$ and $2^A$. So for each element $S$ in $P(A)$, which will be a subset of $A$, we define $\phi(S)$ to be a map $f_S : A \to \mathbb{N}_2$, where $f_S(a) = 1$, if $a \in S$, and $f_S(a) = 2$, if $a \notin S$.

Now we need to check that $\phi$ is a bijection. It’s injective because for any two subset $S_1, S_2$ of $A$, if $\phi(S_1) = f_{S_1} = \phi(S_2) = f_{S_2}$, we can show that $S_1 = S_2$, since then for $a \in S_1$, $f_{S_1}(a) = 1 = f_{S_2}(a)$, so $a \in S_2$, similarly if $a \notin S_1$, $f_{S_1}(a) = 2 = f_{S_2}(a)$, so $a \notin S_2$, so $S_1 = S_2$, $\phi$ is injective. At last, it’s clear that $\phi$ is surjective, since for any map $f : A \to \mathbb{N}$ we can find a subset $S$ of $A$, so that any element $a \in S$ if and only if $f(a) = 1$. So $\phi(S) = f$. It’s a bijection.

Problem (3). Since each $A_n$ is countable, we can denote the elements in $A_n$ as:

$$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\},$$
$$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\},$$
$$A_3 = \{a_{31}, a_{32}, a_{33}, \ldots\},$$
$$\ldots$$
$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}.$$  

Now we can apply the zig-zag argument from the book. We simply enumerates the elements of $A$ along each anti-diagonal of the matrix we have got. In this way we can list elements of $A$ in a sequence $\{a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, \ldots\}$. Of course this sequence may contain duplications, but this sequence has already given us an onto map from natural numbers to $A$. So $A$ has to be finite or countable, while $A$ has infinite subsets $A_n$s, it has to be countable.

Problem (5). We want to rewrite $P_{\text{finite}}(\mathbb{N})$ as a countable union of finite sets. In fact, recall $P(\mathbb{N}_n) = P(\{1, 2, \ldots, n\})$ is the set of all subsets of $\mathbb{N}_n$, which will be a finite set. Clearly,

$$P_{\text{finite}}(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} P(\mathbb{N}_n),$$

so it’s countable by a similar argument as in problem 3.