1. (a) (10 points) State the Min/Max Theorem (also called the Extreme Value Theorem).

The Min/Max theorem states that if $f : [a, b] \to \mathbb{R}$ is continuous, then there are $c, d \in [a, b]$ so that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$. That is, $f$ achieves both its maximum and minimum value on the closed bounded interval $[a, b]$.

(b) (5 points) Give an example of a continuous function $f : (-1, 1) \to \mathbb{R}$ that achieves its maximum value, but does not achieve its minimum value.

An example would be $f(x) = 1 - x^2$. This function achieves its maximum value of $1 = f(0)$ at $x = 0$ but does not achieve a minimum value. This is consistent with a) as the interval is not closed.
(c) (15 points) Show that if \( f : [0, 1] \rightarrow (0, 1) \) is continuous, then \( f \) is not onto.

As \( f \) is continuous and \([0, 1]\) is a closed bounded interval, the Min/Max Theorem implies that there is a value \( c \in [0, 1] \) and a value \( d \in [0, 1] \) so that \( f(c) \leq f(x) \leq f(d) \) for all \( x \in [a, b] \). That is, \( f([0, 1]) \subset [f(c), f(d)] \subset (0, 1) \). By definition, one has \( 0 < f(c) \leq f(d) < 1 \) and so there is some element \( y \in (0, f(c)) \). However, there can be no \( x \in [0, 1] \) so \( f(x) = y \) as one must have \( f(x) \geq f(c) > y \). Hence, \( f \) cannot be onto.
2. (a) (10 points) State the formal definition of uniform continuity of a function \( f : (a, b) \to \mathbb{R} \).

The given function \( f \) is uniformly continuous if, for all \( \epsilon > 0 \), there is a \( \delta > 0 \) so that if \( x, y \in (a, b) \) satisfy \( |x - y| < \delta \), then \( |f(x) - f(y)| < \epsilon \).

(b) (5 points) Give an example of a continuous function \( f : (a, b) \to \mathbb{R} \) that is not bounded. You do not need to justify your answer.

An example with \( (a, b) = (0, 1) \) is \( f(x) = \frac{1}{x} \).
(c) (15 points) Show that if \( f : (a, b) \to \mathbb{R} \) is uniformly continuous, then there is a number \( M \) so \( |f(x)| \leq M \) for all \( x \in (a, b) \). That is, \( f \) is bounded.

As \( f \) is uniformly continuous, there is a \( \delta > 0 \) so that \( x, y \in (a, b) \) with \( |x - y| < \delta \) implies \( |f(x) - f(y)| < 1 \). Choose, \( N \in \mathbb{N} \) so \( \delta N > (b - a) \) and let \( x_i = a + \frac{b - a}{N+1} i \). Observe \( x_i \in (a, b) \) for \( 1 \leq i \leq N \) and \( x_{i+1} - x_i = \frac{b - a}{N+1} \). In particular, for every \( x \in (a, b) \) there is a \( 1 \leq j \leq N \) so \( |x - x_j| < \delta \). Set \( M = \max \{|f(x_1)|, \ldots, |f(x_N)|\} + 1 \).

As already observed, for any \( x \in (a, b) \) there is a \( 1 \leq j \leq N \) so \( |x - x_j| < \delta \). Using the reverse triangle inequality this gives

\[
|f(x)| = |f(x) - f(x_j) + f(x_j)| \leq |f(x) - f(x_j)| + |f(x_j)| \leq 1 + |f(x_j)| \leq M
\]

This proves the claim.
3. (a) (5 points) State the definition of a function \( f : (a, b) \to \mathbb{R} \) being strictly increasing on \((a, b)\).

The function \( f \) is strictly increasing if \( x, y \in (a, b) \) with \( x < y \) satisfies \( f(x) < f(y) \).

(b) (10 points) Show that if \( f : (a, b) \to \mathbb{R} \) is differentiable and \( f'(x) > 0 \) for all \( x \in (a, b) \), then \( f \) is strictly increasing on \((a, b)\).

Pick \( x, y \in (a, b) \) with \( x < y \). By the mean value theorem applied to \([x, y] \subset (a, b)\), there is a \( c \in (x, y) \) so that \( f(y) - f(x) = f'(c)(y - x) \). As \( f'(c) > 0 \) and \( y - x > 0 \) one concludes that \( f(y) - f(x) > 0 \). That is \( f(y) > f(x) \). This means that \( f \) is strictly increasing.
(c) (10 points) Show that if \( f : (a, b) \to \mathbb{R} \) is strictly increasing and \( c \in (a, b) \), then \( \lim_{x \to c^{-}} f(x) \) and \( \lim_{x \to c^{+}} f(x) \) both exist and satisfy \( \lim_{x \to c^{-}} f(x) \leq f(c) \leq \lim_{x \to c^{+}} f(x) \).

Set \( S_- = \{ f(x) : x \in (a, c) \} \). We observe that \( S_- \) is a non-empty set with upper bound \( f(c) \). This is because there are \( y \in (a, c) \) and the monotonicity of \( f \) implies \( f(y) < f(c) \). By the least upper bound property of \( \mathbb{R} \), this means there is a value \( L_- = \sup S_- \leq f(c) \). We claim \( \lim_{x \to c^{-}} f(x) = L_- \). Indeed, given \( \epsilon > 0 \), the definition of least upper bound implies there is a \( y \in S_- \) with \( f(y) > L_- - \epsilon \). The fact that \( f \) is strictly increasing implies that for \( x \in (y, c) \) one has \( L_- - \epsilon < f(y) < f(x) < f(c) \). That is if we set \( \delta = c - y > 0 \) then for any \( x \) so that \( x < c \) and \( 0 < |x - c| < \delta \) one has \( |f(x) - L_-| < \epsilon \). That means \( \lim_{x \to c^-} f(x) = L_- \leq f(c) \).

A similar argument proves that \( \lim_{x \to c^+} f(x) \) exists and is greater than or equal to \( f(c) \).

(d) (10 points) Show that if \( g : (a, b) \to \mathbb{R} \) is differentiable and \( g' : (a, b) \to \mathbb{R} \) is strictly increasing, then \( g' \) is continuous. (Hint: Recall, the derivative of a differentiable function has the intermediate value property.)

By a theorem of Darboux, \( g' \) has the intermediate value property on any interval \([a', b'] \subset (a, b)\). Given a \( c \in (a, b) \) observe that, by the previous problem \( \lim_{x \to c^-} g'(x) = L_- \) and \( \lim_{x \to c^+} g'(x) = L_+ \) both exist and satisfy \( \lim_{x \to c^-} g'(x) \leq f(c) \leq \lim_{x \to c^+} g'(x) \). We claim \( \lim_{x \to c^-} g'(x) = g'(c) = \lim_{x \to c^+} g'(x) \). To see this observe, that if \( L_- < g'(c) \), then any \( d \in (a, c) \), and \( y \in (L_-, g'(c)) \) the intermediate value property of \( g' \) applied to \([d, c]\) implies there is an \( x \in (d, c) \) so \( g'(x) = y \). However, as \( g' \) is strictly increasing and \( a < x < c \) one must have \( g'(x) \leq L_- \). This contradicts \( g'(x) > L_- \) and so one must have \( L_- = g'(c) \). A similar argument shows \( L_+ = g'(c) \).

Hence, \( \lim_{x \to c} g'(x) \) exists and equals \( g'(c) \). That is, \( g' \) is continuous at \( c \) and so \( g' \) is a continuous function.
4. (a) (10 points) Show that if \( f : [a, b] \rightarrow [0, 1] \) satisfies \( f(x) = 0 \) for all \( x \in [a, b] \cap \mathbb{Q} \), then

\[
\int_a^b f(x) \, dx = 0.
\]

That is the lower Darboux integral of \( f \) vanishes.

For any partition \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \), we observe that as \( f(x) \geq 0 \) one has \( m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \geq 0 \). However, as the rationals are dense, one must have \( [x_{i-1}, x_i] \cap \mathbb{Q} \neq \emptyset \). In particular, there is a \( y \in [x_{i-1}, x_i] \) so \( f(y) = 0 \). It follows that \( m_i \leq 0 \) and so \( m_i = 0 \).

Hence,

\[
L(P, f) = \sum_{i=1}^{n} m_i \Delta_i = 0
\]

and so

\[
\int_a^b f(x) \, dx = \sup \{L(P, f) : P \text{ a partition}\} = 0.
\]

(b) (5 points) Give an example of a discontinuous function \( f : [0, 1] \rightarrow \mathbb{R} \) that is Riemann integrable.

The function \( f(x) = \begin{cases} 1 & x \in [0, 1) \\ 0 & x = 1 \end{cases} \) is discontinuous at \( x = 1 \). However, this is the only discontinuity and so \( f \) is Riemann integrable.
(c) (20 points) Let \( f : (a, b) \to \mathbb{R} \) be uniformly continuous. Show directly from definitions that if \( g : [a, b] \to \mathbb{R} \) satisfies \( g(x) = f(x) \) for \( x \in (a, b) \), then \( g \) is Riemann integrable.

As \( f \) is uniformly continuous on the interval \((a, b)\), it is bounded by some value \( M_0 \) as shown (for instance) in a previous question. Set \( M = \max \{M_0, |g(a)|, |g(b)|\} \) one clearly has \(|g(x)| \leq M \) for all \( x \in [a, b] \) and so \( g \) is bounded. In particular, it is enough to show that

\[
\int_a^b g(x)dx = \int_a^b g(x)dx
\]

That is, to show for any \( \epsilon > 0 \) that there is a partition \( P = P_\epsilon \) so that

\[
0 \leq \int_a^b g(x)dx - \int_a^b g(x)dx \leq U(P, f) - L(P, f) \leq \epsilon.
\]

To that end, observe that, as \( f \) is uniformly continuous, for any \( \epsilon > 0 \), there is a \( \delta > 0 \) so that \( x, y \in (a, b) \) with \(|x - y| < \delta\) implies \(|f(x) - f(y)| < \frac{\epsilon}{b-a}\).

Now pick a partition \( P = \{a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b\} \) of \([a, b]\) chosen so \( \Delta_i < \min \{\delta, \frac{\epsilon}{4M}\} \). Set \( M_i = \sup_{x \in [x_{i-1}, x_i]} g(x) \) and \( m_i = \inf_{x \in [x_{i-1}, x_i]} g(x) \). For \( 2 \leq i \leq n \), the definition of \( f \) ensures that \( M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \) and \( m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \) and so, by the uniform continuity of \( f \) and fact that \( x_i - x_{i-1} < \delta \), when \( 2 \leq i \leq n \) one has \( M_i - m_i \leq \frac{\epsilon}{b-a} \).

One readily uses the bound on \( g \) to see that \( M_1 - m_1 \leq 2M \) and \( M_{n+1} - M_n \leq 2M \). Hence,

\[
U(P, f) - L(P, f) = \sum_{i=1}^{n+1} (M_i - m_i) \Delta_i \leq 2M \Delta_1 + 2M \Delta_{n+1} + \frac{\epsilon}{b-a} (x_n - x_1) \leq \epsilon.
\]

As \( \epsilon \) is arbitrary, the claim is proved.
5. (a) (15 points) State both versions of the Fundamental Theorem of Calculus.

The first version of the fundamental theorem of calculus states that if \( F : [a, b] \to \mathbb{R} \) is continuous and differentiable on \((a, b)\) and there is a \( f : [a, b] \to \mathbb{R} \) that is Riemann integrable and so \( F'(x) = f(x) \) for all \( x \in (a, b) \), then

\[
F(b) - F(a) = \int_a^b f(x) \, dx.
\]

The second version states that if \( f : [a, b] \to \mathbb{R} \) is Riemann integrable and \( F(x) = \int_a^x f(t) \, dt \), then \( F \) is continuous. Moreover, if \( f \) is continuous at \( c \in (a, b) \), then \( F \) is differentiable at \( c \) and \( F'(c) = f(c) \).

(b) (5 points) Give an example of a function \( f : [-1, 1] \to \mathbb{R} \) that is Riemann integrable, but \( F(x) = \int_0^x f(t) \, dt \) is not differentiable at \( x = 0 \).

Consider the function

\[
f(x) = \begin{cases} 
-1 & x \in [-1, 0] \\
1 & x \in (0, 1] 
\end{cases}
\]

This function has only one discontinuity (at \( x = 0 \)) so is Riemann integrable. One verifies that \( F(x) = |x| \) and so \( F \) is not differentiable at \( x = 0 \).
(c) (15 points) Show that if \( f : (-1, 1) \to \mathbb{R} \) is \( C^1 \) with \( f(0) = 0 \) and \( f'(x) \geq 2|x| \), then \( |f(x)| \geq x^2 \) for all \( x \in (-1, 1) \).

First observe that there is nothing to show when \( x = 0 \). We treat two cases: \( x \in (0, 1) \) and \( x \in (-1, 0) \). In the first case, the hypotheses on \( f \) (namely that it is \( C^1 \)) allow us to apply the first version of the fundamental theorem of calculus to conclude that

\[
|f(x)| \geq f(x) - f(0) = \int_0^x f'(t)dt \geq \int_0^x 2t = \int_0^x \frac{d}{dt}(t^2)dt = x^2.
\]

In a similar fashion, when \( x \in (-2, 0) \) one has

\[
|f(x)| \geq -f(x) - f(0) = \int_0^x f'(t)dt \geq \int_x^0 (-2t)dt = \int_x^0 \frac{d}{dt}(-t^2)dt = x^2.
\]

Putting these together proves the claim.
6. (a) (10 points) State the definition of a sequence of functions $f_n : [a, b] \to \mathbb{R}$ uniformly converging to $f : [a, b] \to \mathbb{R}$.

The sequence converges uniformly on $[a, b]$ if, for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that if $n \geq N$, then $\sup_{x \in [a, b]} |f(x) - f_n(x)| < \epsilon$.

(b) (5 points) Give an example of a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ so that $f_n$ converges pointwise to $f : [0, 1] \to \mathbb{R}$ but not uniformly.

An example is given by $f_n(x) = x^n$. One has $\lim_{n \to \infty} x^n = 0$ if $x \in [0, 1)$ while $\lim_{n \to \infty} x^n = 1$ for $x = 1$ and so $f_n$ converge pointwise to the function

$$f(x) = \begin{cases} 
0 & x \in [0, 1) \\
1 & x = 1
\end{cases}$$

However, the convergence cannot be uniform (as for instance the uniform limit of continuous functions is continuous and $f$ is not.)
(c) (20 points) Prove that if \( f_n : [a, b] \to \mathbb{R} \) are continuous and the \( f_n \) converge uniformly to \( f : [a, b] \to \mathbb{R} \), then \( f \) is continuous.

Fix an \( \epsilon > 0 \). As \( f_n \to f \) uniformly, there is an \( N \in \mathbb{N} \) so that for \( n \geq N \), \( \sup_{x \in [a, b]} |f(x) - f_n(x)| < \epsilon/3 \). Now fix an \( c \in [a, b] \). As \( f_N \) is continuous, there is a \( \delta > 0 \) so for all \( x \in [a, b] \) with \( |x - c| < \delta \) one has \( |f_N(x) - f_N(c)| < \epsilon/3 \). Using the triangle inequality, one then concludes that, for \( x \in [a, b] \) with \( |x - c| < \delta \), one has

\[
|f(x) - f(c)| = |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\
\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

That is, \( f \) is continuous at \( x = c \). As \( c \) is arbitrary, one concludes that \( f \) is continuous.