Solutions Final Exam — May. 14, 2014

1. Determine whether the following statements are true or false. Justify your answer (i.e., prove the claim, derive a contradiction or give a counter-example).

(a) (10 points) There exist open intervals $I_n$ with $I_{n+1} \subset I_n$ so that $\cap_{n=1}^{\infty} I_n = \emptyset$.

True. Let $I_n = (0, 1/n)$. If $z \in \cap_{n=1}^{\infty} I_n$, then $0 < z < 1/n$ for all $n$, which violates the Archimedean principle.

(b) (10 points) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

True. Given $\epsilon > 0$, use the uniform continuity of $f$ to pick $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Now use the Cauchy property of $\{x_n\}$ to pick an $N$ so that if $N < m, n$, then $|x_m - x_n| < \delta$. Hence, if $N < m, n$, then $|f(x_n) - f(x_m)| < \epsilon$, i.e., $\{f(x_n)\}$ is Cauchy.
(c) (10 points) If \( f : (a, b) \to \mathbb{R} \) is \( C^1 \) and strictly increasing, then \( f'(x) > 0 \) for each \( x \in (a, b) \).

False. Let \((a, b) = (-1, 1)\) and \( f(x) = x^3 \), then \( x < y \) implies \( f(x) < f(y) \), but \( f'(0) = 0 \).

(d) (10 points) If \( f : (-1, 1) \to \mathbb{R} \) is \( C^2 \) with \( f(0) = f'(0) = 0 \) and \( f''(0) = 2 \), then there is an interval \( I \) containing 0 so that \( f(x) \geq 0 \) for \( x \in I \).

True. By Taylor’s theorem \( f(x) = \frac{1}{2}f''(0)x^2 + o(x^2) = x^2 + o(x^2) \). Choose, \( \epsilon > 0 \) so that if \( |x| < \epsilon \), then \( |f(x) - x^2| < \frac{1}{2}|x|^2 \). By the triangle inequality this means that \( f(x) > \frac{1}{2}|x|^2 \geq 0 \) for \( x \in (-\epsilon, \epsilon) \).
(e) (10 points) If $\lim_{n \to \infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

\begin{center}
False. Let $a_n = \frac{1}{n}$, this series has $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$.
\end{center}

(f) (10 points) There is a sequence of continuous functions $f_n : [-1, 1] \to \mathbb{R}$ converging uniformly to the function $f : [-1, 1] \to \mathbb{R}$ given by $f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$.

\begin{center}
False. For $f$ to be the uniform limit of continuous functions, it must itself be continuous. The given $f$ is not continuous as $\lim_{x \to 0} f(x)$ does not exist.
\end{center}
(g) (10 points) If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then for all bijections $m : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^{\infty} a_{m(n)}$ is convergent and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{m(n)}$.

True. Consider partitions $P_n = \left\{ 0 < \frac{1}{n} < \cdots < \frac{k}{n} < \cdots < 1 \right\}$. Then $|P_n| \to 0$, and so $S(f, P_n, A) \to \int_0^1 f(x)dx$ for any choice of $A$, picking $A$ to be the left endpoints, we see that $S(f, P_n, A) = 0$ and so $\int_0^1 f(x)dx = 0$.

(h) (10 points) Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable. If $f(q) = 0$ for all rational numbers $q \in [0, 1]$, then $\int_0^1 f(x)dx = 0$.

True. Consider partitions $P_n = \left\{ 0 < \frac{1}{n} < \cdots < \frac{k}{n} < \cdots < 1 \right\}$. Then $|P_n| \to 0$, and so $S(f, P_n, A) \to \int_0^1 f(x)dx$ for any choice of $A$, picking $A$ to be the left endpoints, we see that $S(f, P_n, A) = 0$ and so $\int_0^1 f(x)dx = 0$. 

False. If the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ convergent. However, there is a rearrangement $m : \mathbb{N} \to \mathbb{N}$ so that $\sum_{n=1}^{\infty} (-1)^{m(n)} \frac{1}{m(n)}$ diverges to infinity. 


2. (a) (10 points) State the intermediate value theorem.

Let \( f : [a, b] \to \mathbb{R} \) be continuous with \( f(a) < f(b) \), then for every \( y \in [f(a), f(b)] \), there is an \( x \in [a, b] \) so that \( y = f(x) \).

(b) (15 points) Show that if \( f : \mathbb{R} \to \mathbb{R} \) is continuous and \( I \subseteq \mathbb{R} \) is a compact interval, then \( f(I) \) is compact interval.

Write \( I = [a, b] \). Set \( M = \sup_I f(x) \) and \( m = \inf_I f(x) \). As \( I \) is compact and \( f \) continuous, there are \( x, y \in I \) with \( f(x) = m \) and \( f(y) = M \). Hence, \( M, m \in \mathbb{R} \). We have \( \{m, M\} \subseteq f(I) \subseteq [m, M] \) and so if \( m = M \) we have nothing to prove. If \( m < M \), then either \( x < y \) or \( y < x \). WLOG we assume \( x < y \). For any \( z \in (m, M) \) there is a \( z' \in (x, y) \subseteq I \) with \( f(z') = z \). This implies that \( f(I) \supset [m, M] \), that is \( f(I) = [m, M] \).
3. (a) (10 points) State the mean value theorem.

Let \( f : (a, b) \to \mathbb{R} \) be differentiable. For \( a < x < y < b \), there is a \( z \in (x, y) \) so that
\[
f'(z) = \frac{f(y) - f(x)}{y - x}.
\]

(b) (15 points) Show that if \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and \( f'(x) \geq x \), then \( f(x) \leq f(0) + \frac{1}{2}x^2 \) when \( x \leq 0 \).

Consider the function \( g(x) = f(x) - f(0) - \frac{1}{2}x^2 \). This function is also differentiable as \( f(0) + \frac{1}{2}x^2 \) is. One computes that \( g'(x) = f'(x) - x > 0 \), so \( g \) is non-decreasing. Moreover, \( g(0) = 0 \). Hence, \( g(x) \leq 0 \) for \( x \leq 0 \). Which proves the claim.
4. (a) (10 points) State one of the (equivalent) definitions of a function $f : [a, b] \to \mathbb{R}$ being Riemann integrable.

$f$ is Riemann integrable if it is bounded and for every $\epsilon > 0$, there is a $\delta > 0$, so that if $P$ is a partition with $|P| < \delta$, then $Osc(f, P) = S^+(f, P) - S^-(f, P) < \epsilon$.

(b) (10 points) Give an example of a function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable. You do not need to justify this.

Consider Dirichlet’s function $f(x) = \begin{cases} 
1 & x \text{ rational} \\
0 & x \text{ irrational} 
\end{cases}$
(c) (15 points) Using the definition from a), show that if \( f : [a, b] \to \mathbb{R} \) is continuous, then it is Riemann integrable.

As \( f \) is continuous and \([a, b]\) is compact, \( f \) is uniformly continuous and is bounded. Using the uniform continuity of \( f \), given an \( \epsilon > 0 \), pick \( \delta > 0 \) so that \( |x - y| < \delta \) implies \( |f(x) - f(y)| < \frac{\epsilon}{b-a} \). For any partition, \( P = \{a = x_0 < x_1 < \ldots < x_n = b\} \) we have \( S^+(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \) where \( M_i = \sup_{[x_{i-1}, x_i]} f(x) \) and \( S^-(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \) where \( m_i = \inf_{[x_{i-1}, x_i]} f(x) \). By the continuity of \( f \) and compactness of \([x_{i-1}, x_i]\) we have \( M_i = f(a_i) \) and \( m_i = f(b_i) \). Hence, if \( |P| < \delta \), then \( \text{Osc}(f, P) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \epsilon \) as \( |a_i - b_i| < \delta \).
5. Let \( f : D \to \mathbb{R} \) be a function.

(a) (10 points) State the definition of \( f \) being (real) analytic.

\[
f \text{ is real analytic, if } D \text{ is open and for every } x_0 \in D, \text{ there is a power series } \sum_{n=0}^{\infty} a_n(x_0)(x-x_0)^n \text{ with positive radius of converge and so that } f(x) = \sum_{n=0}^{\infty} a_n(x_0)(x-x_0)^n \text{ in a neighborhood of } x_0.
\]

(b) (10 points) Give an example of a function \( f \) that is infinitely differentiable (i.e. of class \( C^\infty \)) but that is not real analytic. You do not need to justify your answer.

Let \( f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases} \). This function is infinitely differentiable, but is not real analytic as \( f \) does not agree with any powerseries near \( x = 0 \).
(c) (15 points) Show that if \( D \) is an interval, \( f \) is real analytic and \( f(x) = 0 \) for all \( x \in I \) for \( I \subset D \) an open interval, then \( f(x) = 0 \) for all \( x \in D \).

**Hint:** Consider the maximum interval containing \( I \) on which \( f \) vanishes. Using the Taylor polynomials at the endpoints prove this interval is \( D \).

Set \( D = (a, b) \) and \( I = (c, d) \). Let \( z_- = \inf \{ x : f(z) = 0 \text{ for all } z \in (x, d) \} \) and set \( z_+ = \sup \{ x : f(z) = 0 \text{ for all } z \in (c, x) \} \). If \( z_- = a \) and \( z_+ = b \), then there is nothing to prove.

Assume \( z_- \neq a \) – that is, \( z_- \in D \). There is a sequence \( x_n \in (z_-, d) \) with \( x_n \to z_- \). Notice, that as \( f(x) = 0 \) in all of \((z_-, d)\) that \( f^{(n)}(x_k) = 0 \) for all \( n \). As \( f \) is analytic, it is \( C^n \) for all \( n \). Hence, passing to a limit and using the continuity of \( f^{(n)} \) we see that \( f^{(n)}(z_-) = 0 \) – we use here that \( z_- \in D \). As \( f \) is analytic and \( z_- \in D \), there is an interval \( R > 0 \) so that \( f(x) = \sum_{n=0}^{\infty} a_n (x - z_-)^n \) when \( |x - z_-| < R \) and \( x > a \). However, as \( f^{(n)}(z_-) = 0 \) for all \( n \) we have that \( a_n = 0 \) for all \( n \). Hence, for any \( x \) with \( |x - z_-| < R \) and \( x > a_- \) we must have \( f(x) = 0 \). However, this contradicts our definition of \( z_- \) proving that \( z_- = a \). That \( z_+ = b \) is proved in the same way.