Mathematic 405, Fall 2015: Assignment #5

Due: Wednesday, March 11th

Instructions: Please ensure that your answers are legible. Also make sure that sufficient steps are shown. Page numbers refer to the course text.

Problem #1. Use the intermediate value theorem to show that if \( p(x) = \sum_{i=1}^{n} a_i x^i \) is a degree \( n \) polynomial (so \( a_n \neq 0 \)) and \( n \) is odd, then \( p \) must have a real zero.

Problem #2. Let \( f : [0, 1] \to [0, 1] \) be continuous. Use the intermediate value theorem to show that \( f \) has at least one fixed point – i.e., a point satisfying \( f(x) = x \).

Problem #3. Show that if \( f : (0, 1) \to \mathbb{R} \) is uniformly continuous, then \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 1^-} f(x) \) both exist. Use this to show that there is a uniformly continuous function \( \hat{f} : [0, 1] \to \mathbb{R} \) with \( \hat{f}(x) = f(x) \) for all \( x \in (0, 1) \). Give an example to show this is not possible if \( f \) is only continuous.

Problem #4. p. 125 # 6

Problem #5. p. 125 # 7

Problem #6. p. 138 # 10

Problem #7. p. 138 # 11

Problem #8. p. 138 # 12

Bonus Problem. (Will not be graded)

Let \( C \subset \mathbb{R} \) be an arbitrary non-empty compact set and suppose \( f : C \to C \) satisfies \( |f(x) - f(y)| \leq \alpha |x - y| \) for all \( x, y \in C \) and some \( \alpha \in (0, 1) \) (in particular \( f \) is Lipschitz). Such a map is an example of a contraction.

a) Show that \( f \) has a fixed point. (Hint: Show that the inductively defined sequence \( a_1 = x_1, a_{n+1} = f(a_n) \) is Cauchy where here \( x_0 \) an arbitrary point of \( C \)).

b) Show that this fixed point is the only fixed point of \( f \).

c) What happens if \( C \) is not compact or \( f \) is merely continuous.