Math 405 Final Exam — May. 13, 2015

Name: ______________________________________

- Complete the following problems. In order to receive full credit, please make sure to *justify your answers*. You are free to use results from class or the course textbook as long as you clearly state what you are citing.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted (nor are they needed). If you finish early, you must hand your exam paper to a proctor.
- Please check that your copy of this exam contains 12 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

The following boxes are strictly for grading purposes. Please do not mark.

Question	Points	Score
1	30	
2	30	
3	30	
4	40	
5	30	
6	40	
Total	200	

1. (a) (10 points) State the formal definition of a Cauchy sequence of real numbers.

(b) (5 points) Give an example of a sequence of real numbers, $\{a_n\}_{n\in\mathbb{N}}$, which satisfies $\lim_{n\to\infty} |a_{n+1} - a_n| \to 0$, but which is *not* Cauchy. You do not need to justify your answer.

(c) (15 points) Arguing directly from the definition, show that if both $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are Cauchy, then so is the sequence $\{a_nb_n\}_{n\in\mathbb{N}}$.

2. (a) (10 points) State the formal definition of a compact subset (of $\mathbb{R}$).

(b) (5 points) Give an example of a non-compact set A and a continuous function $f : A \to \mathbb{R}$ so that there is no $x_0 \in A$ so that $f(x_0) \geq f(x)$ for all $x \in A$ – i.e., f does *not* achieve its maximum. You do not need to justify your answer.

(c) (15 points) Show that if $A \subset \mathbb{R}$ is compact and non-empty and $f : A \to \mathbb{R}$ is continuous, then there is a value $x_0 \in A$ so that $f(x_0) \geq f(x)$ for all $x \in A$.

3. (a) (10 points) State the mean value theorem.

   (b) (5 points) Give an example of a function $f : (-1, 1) \to \mathbb{R}$ with the property that there is no differentiable function $F : (-1, 1) \to \mathbb{R}$ with $F' = f$. You do not need to justify your answer.

(c) (15 points) Show that if $f : (a,b) \to \mathbb{R}$ is differentiable and $\sup_{x\in(a,b)} |f'(x)| < C$, then for all $x, y \in (a,b)$, $|f(x) - f(y)| \le C|x-y|$.

4. (a) (10 points) State one of the (equivalent) definitions of a function $f : [a, b] \to \mathbb{R}$ being Riemann integrable.

(b) (10 points) Give an example of a function $f : [0, 1] \to \mathbb{R}$ which is *not* Riemann integrable. You do not need to justify your answer.

(c) (20 points) Using the definition from (a) directly, show that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is Riemann integrable.

5. (a) (15 points) State both directions of the fundamental theorem of calculus

(b) (5 points) Give a Riemann integrable function, $f : [-1, 1] \to \mathbb{R}$, for which the function $F(x) = \int_0^x f(t)dt$ is not differentiable at some point of $(-1, 1)$. You do not need to justify your answer.

(c) (10 points) Suppose $f, g : (a, b) \to \mathbb{R}$ are C^1 and that $[c, d] \subset (a, b)$. Show that

$$\int_c^d f'(x)g(x)dx = f(d)g(d) - f(c)g(c) - \int_c^d f(x)g'(x)dx.$$

6. (a) (10 points) Fix an interval $I \subset \mathbb{R}$ and let $f_n : I \to \mathbb{R}$, $n \in \mathbb{N}$, and $f : I \to \mathbb{R}$ be functions. State the definition of f_n converging uniformly to f.

(b) (10 points) Give an example of a power series $\sum_{n=0}^{\infty} a_n x^n$ which converges pointwise on $(-1, 1)$ but not uniformly. You do not need to justify your answer.

(c) (20 points) Fix an interval $I \subset \mathbb{R}$ and let $f_n : I \to \mathbb{R}$, $n \in \mathbb{N}$, be functions which satisfy

1. for all $x \in I$ and $n \in \mathbb{N}$, $0 \le f_{n+1}(x) \le f_n(x)$, and
2. for all $x \in I$, $\lim_{n\to\infty} \sup_{x \in I} f_n(x) = 0$.

Show that the series $\sum_{n=1}^{\infty}(-1)^n f_n(x)$ converges uniformly on I. Hint: show that for $m > N$:

$$0 \le (-1)^N \sum_{k=N}^{m} (-1)^k f_k(x) \le f_N(x).$$

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