1. We will show that the max function $M(x)$ is continuous. The case for min $m(x)$ is similar. Let $x \in S$. Suppose first that $f(x) > g(x)$. Then by Problem 7 in PS5, there is $\delta > 0$ such that $|y - x| < \delta$ implies $f(y) - g(y) > 0$. Consequently if $|y - x| < \delta$ we have $M(y) = f(y)$, and it follows that $M$ is continuous at $x$ since $f$ is continuous. Similarly we can deal with the case when $g(x) > f(x)$.

Thus it remains to consider the case when $f(x) = g(x) = L$. Let $\varepsilon > 0$. Since $f$ and $g$ are both continuous, there are $\delta_1$, $\delta_2$ respectively such that $|y - x| < \delta_1 \implies |f(y) - L| < \varepsilon$ and $|y - x| < \delta_2 \implies |g(y) - L| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. It follows that if $|y - x| < \delta$ we have

$$|M(y) - M(x)| = |M(y) - L| = \begin{cases} |g(y) - L| & g(y) > f(y) \\ |f(y) - L| & g(y) \leq f(y) < \varepsilon \end{cases}$$

So $M$ is continuous at $x$.

2. Consider $g(x) = f(x) - x$. Then $g(0) = f(0) \geq 0$. If $g(0) = 0$ then $f(0) = 0$ and we are done since we can take $c = 0$. So we may assume $g(0) > 0$. Moreover $g(1) = f(1) - 1 \leq 0$. If $g(1) = 0$ then we are also done since $f(1) = 1$, so we may assume $g(1) < 0$. But then intermediate value theorem implies there is $c \in (0, 1)$ such that $g(c) = 0 \implies f(c) = c$.

3. (a) Take for example $f(x) = \frac{1}{2}$ for $x \in [0, 1]$.

(b) Suppose for a contradiction that $f$ is onto. Let $x_n \in [0, 1]$ be such that $f(x_n) = \frac{1}{n}$. By Bolzano-Weierstrass we may extract a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to x \in [0, 1]$. Since $f$ is continuous it follows that

$$0 = \lim_{k \to \infty} \frac{1}{n_k} = \lim_{k \to \infty} f(x_{n_k}) = f(x)$$

which is a contradiction since 0 is not in the range of $f$.

4. (a) We show that $f$ is discontinuous at 0. Let $\varepsilon = \frac{1}{2}$. For any $\delta > 0$ choose $n \in \mathbb{N}$ large enough so that

$$x = \frac{1}{2n\pi + \frac{\pi}{2}} < \delta$$

and it follows that

$$f(x) = \sin(2n\pi + \frac{\pi}{2}) = 1 \implies |f(x) - f(0)| = 1 > \frac{1}{2}$$

so no such $\delta$ in the definition of continuity exists. This means $f$ is discontinuous at 0.

(b) $f$ is continuous away from 0 as a composition of continuous function, so $f$ automatically has the intermediate value property if $0 < a < b$ or $a < b < 0$. It remains to verify the property when $a < 0 < b$. Let $y$ be such that $f(a) < y < f(b)$. Let $\theta \in [-\pi/2, \pi/2]$ be the unique angle such that $\sin(\theta) = f(a)$. Choose $n \in \mathbb{N}$ large enough such that

$$0 < a' = \frac{1}{2n\pi + \theta} < b$$

Thus we have found a point $a' \in \mathbb{R}$ with $0 < a' < b$ and $f(a') = f(a)$. Intermediate value theorem now implies that there is $c \in (a', b) \subset (a, b)$ such that $f(c) = y$.

5. (a) When $n$ is odd, $p_A(t)$ is an odd degree polynomial. This means that we can find $t_1 < 0$ such that $p_A(t_1) < 0$ (since the leading term is odd-degree), and $t_2 > 0$ such that $p_A(t_2) > 0$. Intermediate value theorem implies there is some $t_0 \in (t_1, t_2)$ such that $p_A(t_0) = 0$.

(b) When $n$ is even we can find $t_1 < 0$ such that $p_A(t_1) > 0$ and $t_2 > 0$ such that $p_A(t_2) > 0$. Since $p_A(0) < 0$, intermediate value theorem implies that we can find $t_0 \in (t_1, 0)$ and $t'_0 \in (0, t_2)$ such that $p_A(t_0) = p_A(t'_0) = 0$.

6. (a) Consider $f(x) = \frac{1}{x}$ and $x_n = \frac{1}{n}$. Clearly $\{x_n\}_{n=1}^{\infty}$ is Cauchy, but $\{f(x_n)\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$ is clearly not Cauchy.
(b) By Theorem 2.4.5 in the notes, a sequence of real number is Cauchy if and only if it converges. Hence we may assume \( x_n \to x \in \mathbb{R} \) as \( n \to \infty \). Since \( g \) is continuous we have \( g(x_n) \to g(x) \), so \( \{g(x_n)\}_{n=1}^{\infty} \) is also Cauchy.

7. For \( h > 0 \) we consider the difference quotient

\[
\left| \frac{f(h) - f(0)}{h - 0} \right| = \frac{f(h) - f(0)}{h} = \begin{cases} h \quad h \in \mathbb{Q} \\ 0 \quad h \not\in \mathbb{Q} \end{cases}
\]

In both cases as \( h \to 0 \) we have

\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = 0
\]

so \( f \) is differentiable at \( x = 0 \). To show that \( f \) is discontinuous everywhere else, we first consider \( x \in \mathbb{Q} \). Then \( f(x) = x^2 \neq 0 \). Let \( \varepsilon = \frac{x^2}{2} \). For any \( \delta > 0 \) choose \( y \in (x - \delta, x + \delta) \setminus \mathbb{Q} \) (such \( y \) exists since \( \mathbb{Q} \) is countable). It follows that

\[
|f(x) - f(y)| = x^2 > \frac{x^2}{2}
\]

so \( f \) is not continuous at \( x \). If \( x \in \mathbb{R} \setminus \mathbb{Q} \). Let \( \varepsilon = \frac{x^2}{8} \), and given \( \delta > 0 \) choose \( y \in (x - \min\{\delta, |x|/2\}, x + \min\{\delta, |x|/2\}) \cap \mathbb{Q} \) (such \( y \) exists since \( \mathbb{Q} \) is dense in \( \mathbb{R} \)). It follows that

\[
|f(x) - f(y)| = y^2 > \frac{x^2}{4} > \frac{x^2}{8}
\]

so \( f \) is not continuous at \( x \) either.

8. (a) Clearly \( f \) is differentiable away from 0 since the function is a product of differentiable function. At \( x = 0 \), for \( h > 0 \) we consider

\[
\left| \frac{f(h) - f(0)}{h - 0} \right| = \left| \frac{h^2 \sin(1/h)}{h} \right| = |h \sin(1/h)|
\]

Note that

\[
\lim_{h \to 0} |h \sin(1/h)| = \lim_{x \to \infty} \frac{\sin(x)}{x} = 0
\]

by squeeze theorem. So we have

\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = 0
\]

meaning \( f \) is differentiable at 0 with \( f'(0) = 0 \).

(b) By the chain rule we compute directly that away from 0

\[
f'(x) = 2x \sin(1/x) - \cos(1/x)
\]

If \( f'(x) \) were continuous at \( x = 0 \) we would have

\[
\lim_{x \to 0} 2x \sin(1/x) - \cos(1/x) = f'(0) = 0
\]

but by Problem 4 we see that the limit \( \cos(1/x) \) as \( x \to 0 \) does not exist, so \( f'(x) \) is not continuous at 0.