1. Let \( \varepsilon > 0 \). Since \( f \) is Riemann integrable, there is a partition \( P_1 = \{x_0, \ldots, x_{n_1}\} \) such that

\[
U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2}
\]

(This follows by choosing a partition \( Q_1 \) such that \( U(Q_1, f) - \inf_P U(P, f) < \varepsilon/2 \) and a partition \( Q_2 \) such that \( \sup_P L(P, f) - L(Q_2, f) < \varepsilon/2 \), then take a common refinement of \( Q_1 \) and \( Q_2 \).) This means for any \( x_k^* \in [x_{k-1}', x_k'] \) we have

\[
U(P_1, f) - \sum_{k=1}^{n_1} f(x_k^*) \Delta x_k < U(P, f) - L(P, f) < \frac{\varepsilon}{2}
\]

Since \( f \) is Riemann integrable there is another partition \( P_2 \) such that

\[
\left| U(P_2, f) - \int_a^b f(x) \, dx \right| < \frac{\varepsilon}{2}
\]

Let \( P = \{x_0, \ldots, x_n\} \) be a common refinement of \( P_1 \) and \( P_2 \), then we have by triangle inequality

\[
\left| \int_a^b f(x) \, dx - \sum_{k=1}^n f(x_k^*) \Delta x_k \right| < \left| \int_a^b f(x) \, dx - U(P, f) \right| + \left| U(P, f) - \sum_{k=1}^n f(x_k^*) \Delta x_k \right| < \varepsilon
\]

2. Since \( f \) is continuous on a bounded interval, we can bound \( f \) by \( f(x_1) \leq f(x) \leq f(x_2) \) for some \( x_1, x_2 \in [a, b] \). Therefore appealing to Riemann sum we have

\[
f(x_1)(b - a) \leq \int_a^b f(x) \, dx \leq f(x_2)(b - a)
\]

Intermediate value theorem applied to the function \( g(x) = f(x)(b - a) \) says that there is \( c \in [x_1, x_2] \) (or \( [x_2, x_1] \), depending on the order) such that

\[
f(c)(b - a) = \int_a^b f(x) \, dx
\]

3. Split the interval as \([a, c]\) and \([c, b]\). Consider \( c_n = c - \frac{1}{n} \) (for \( n \) sufficiently large). Then since \( f = g \) on \([a, c_n]\) we have that \( g \) is integrable on \([a, c_n]\) with

\[
\int_a^{c_n} f(x) \, dx = \int_a^{c_n} g(x) \, dx
\]

Now by Lemma 5.2.8 we may pass to the limit to conclude that \( g \) is integrable on \([a, c]\) and that

\[
\int_a^c g(x) \, dx = \lim_{n \to \infty} \int_a^{c_n} g(x) \, dx = \lim_{n \to \infty} \int_a^{c_n} f(x) \, dx = \int_a^c f(x) \, dx
\]

Doing the same thing on \([c, b]\) and adding up the integral gives the required result.

4. Since \( f \) is continuous, so is \(|f|\), so we have that \(|f|\) is Riemann integrable and that (since \( f \leq |f| \))

\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx = \int_a^b |f(x)| \, dx
\]

When \( f \) is only Riemann integrable, the problem becomes much harder since we do not know if \(|f|\) is Riemann integrable (so one has to work directly with Riemann sums to prove that \(|f|\) is Riemann integrable).

5. Since \( f \) is monotonically increasing on \([a, b]\) we have that \( f(a) \leq f(x) \leq f(b) \) for \( x \in [a, b] \). Now let \( \varepsilon > 0 \), and consider a uniform partition \( P_n \) of \([a, b]\) (so that \( x_0 = a, x_1 = a + \frac{b-a}{n}, \ldots \)). Then we have by monotonicity

\[
U(P_n, f) - L(P_n, f) = \sum_{k=1}^n (f(x_k) - f(x_{k-1})) \frac{b-a}{n} \leq \frac{b-a}{n} \sum_{k=1}^n f(x_k) - f(x_{k-1}) = \frac{b-a}{n} (f(b) - f(a))
\]

since the sum telescopes. It follows that if we choose \( n \) sufficiently large we have \( U(P_n, f) - L(P_n, f) < \varepsilon \), so \( f \) is Riemann integrable.
6. (a) First we show that $f(x) = 0$ on $(a, b)$. Suppose for a contradiction that there is some $x_0 \in (a, b)$ such that $f(x_0) \neq 0$. Without loss of generality we may assume $f(x_0) > 0$. Since $f$ is continuous, there is $\delta > 0$ such that $|x - x_0| < \delta \implies f(x) > \frac{f(x_0)}{2}$. This means that

$$
\int_a^b f(x)^2 \, dx \geq \int_{x_0 - \delta}^{x_0 + \delta} f(x)^2 \, dx > 2\delta \frac{f(x_0)^2}{4} > 0
$$

a contradiction.

Now suppose again for a contradiction that $f(a) > 0$. The same idea works except one can only use a one-sided interval, i.e. there is $\delta > 0$ such that $x - a < \delta \implies f(x) > \frac{f(a)}{2}$. The rest is very similar.

(b) We first show that $f(x) = 0$ on $(a, b)$. Suppose for a contradiction (as above) that there is $x_0 \in (a, b)$ such that $f(x_0) > 0$. Again there is $\delta$ such that $|x - x_0| < \delta \implies f(x) > \frac{f(x_0)}{2}$.

Let $\phi$ be a function that agrees with $f$ on $(x_0 - \delta/2, x_0 + \delta/2)$, linear on $(x_0 - \delta, x_0 - \delta/2)$ and $(x_0 + \delta/2, x_0 + \delta)$ with $\phi(x_0 - \delta) = \phi(x_0 + \delta) = 0$, and 0 elsewhere. It is easy to see that $\phi$ is continuous, and that

$$
\int_a^b f(x)\phi(x) \, dx \geq \int_{x_0 - \delta/2}^{x_0 + \delta/2} f(x)^2 \, dx > \delta \frac{\phi(x_0)^2}{4} > 0
$$

a contradiction.

Now suppose for a contradiction that $f(a) > 0$. Again we find $\delta$ such that $x - a < \delta \implies f(x) > \frac{f(a)}{2}$. Let $\phi$ be a function that agrees with $f$ on $[a + \delta/4, a + \delta/2]$, linear on $[a, a + \delta/4]$ and $(a + \delta/2, a + \delta]$ with $\phi(a) = \phi(a + \delta) = 0$, and 0 elsewhere. Again $\phi$ is continuous and

$$
\int_a^b f(x)\phi(x) \, dx \geq \int_{a + \delta/4}^{a + \delta/2} f(x)^2 \, dx > \frac{\delta}{4} \frac{f(a)^2}{4} > 0
$$

a contradiction. So $f$ must be identically 0.

7. This follows from the fundamental theorem of calculus and the product rule, i.e.

$$
\int_a^b F(x)G'(x) + F'(x)G(x) \, dx = \int_a^b (F(x)G(x))' \, dx = F(b)G(b) - F(a)G(a)
$$

8. By the fundamental theorem of calculus we have for $x \in (0, 2)$,

$$
f(x) - f(0) = \int_0^x f'(y) \, dy \geq \int_0^x y \, dy = \frac{1}{2} x^2
$$

which implies $f(x) \geq \frac{1}{2} x^2$ since $f(0) = 0$.

On the other hand for $x \in (-2, 0]$ we have

$$
f(0) - f(x) = \int_x^0 f'(y) \, dy \geq \int_x^0 y \, dy = -\frac{1}{2} x^2
$$

So we get $f(x) \leq \frac{1}{2} x^2$ instead.